Geometrical Sample Reweighting for Monte Carlo Integration

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PG20+21
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Monte Carlo Integration

The standard solution for offline rendering

$$\hat{I} = \frac{1}{N} \sum_{i=1}^{N} f(x_i)$$

Monte Carlo Integration

The standard solution for offline rendering

$$E[\hat{I}] = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} f(x_i) = \int f(x) \mathrm{d}x$$

Monte Carlo Integration

The standard solution for offline rendering

> hours

Monte Carlo Integration

Monte Carlo Integration

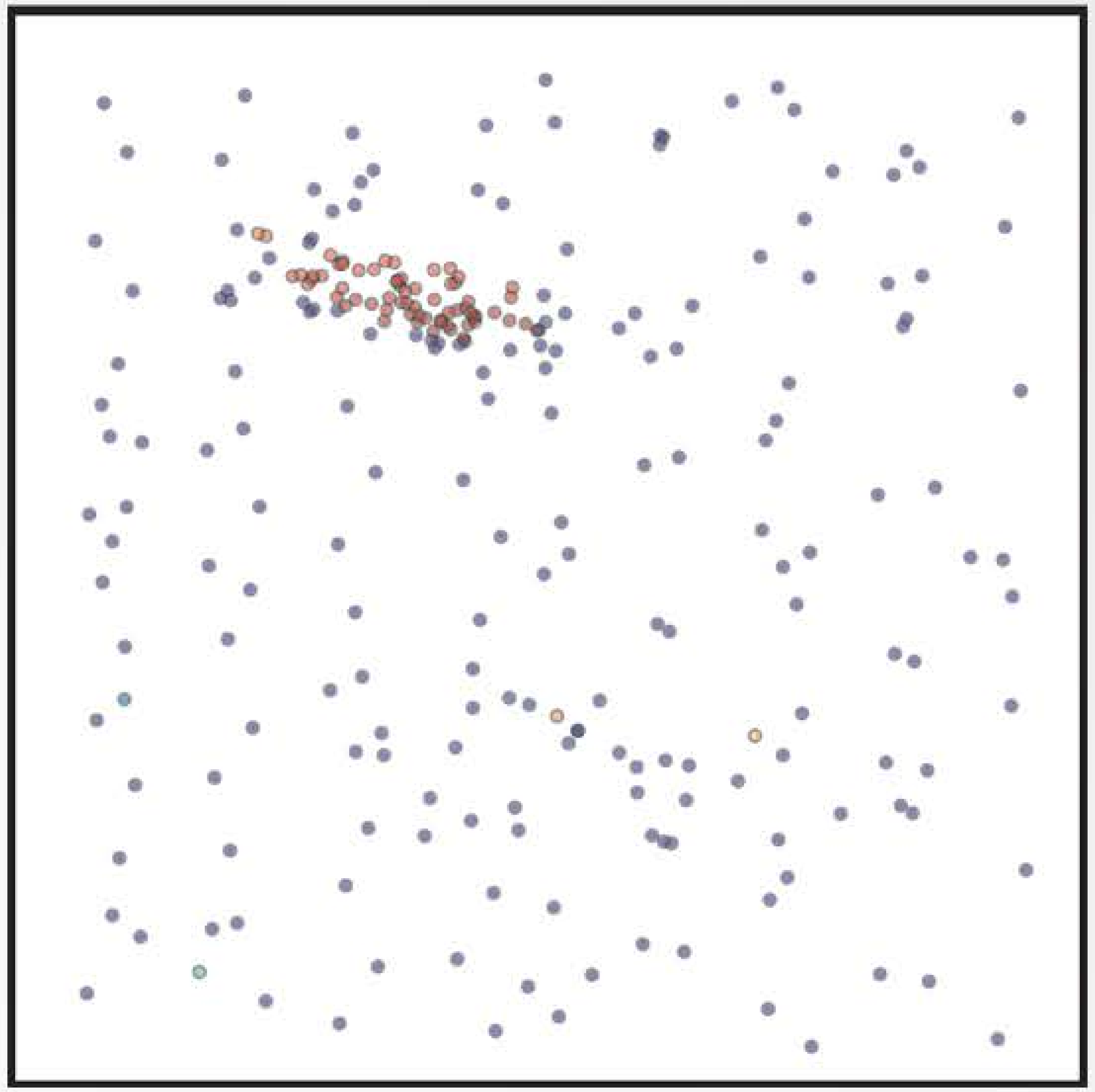
Variance reduction

- Importance sampling
- Quasi Monte Carlo

Monte Carlo Integration

Importance sampling

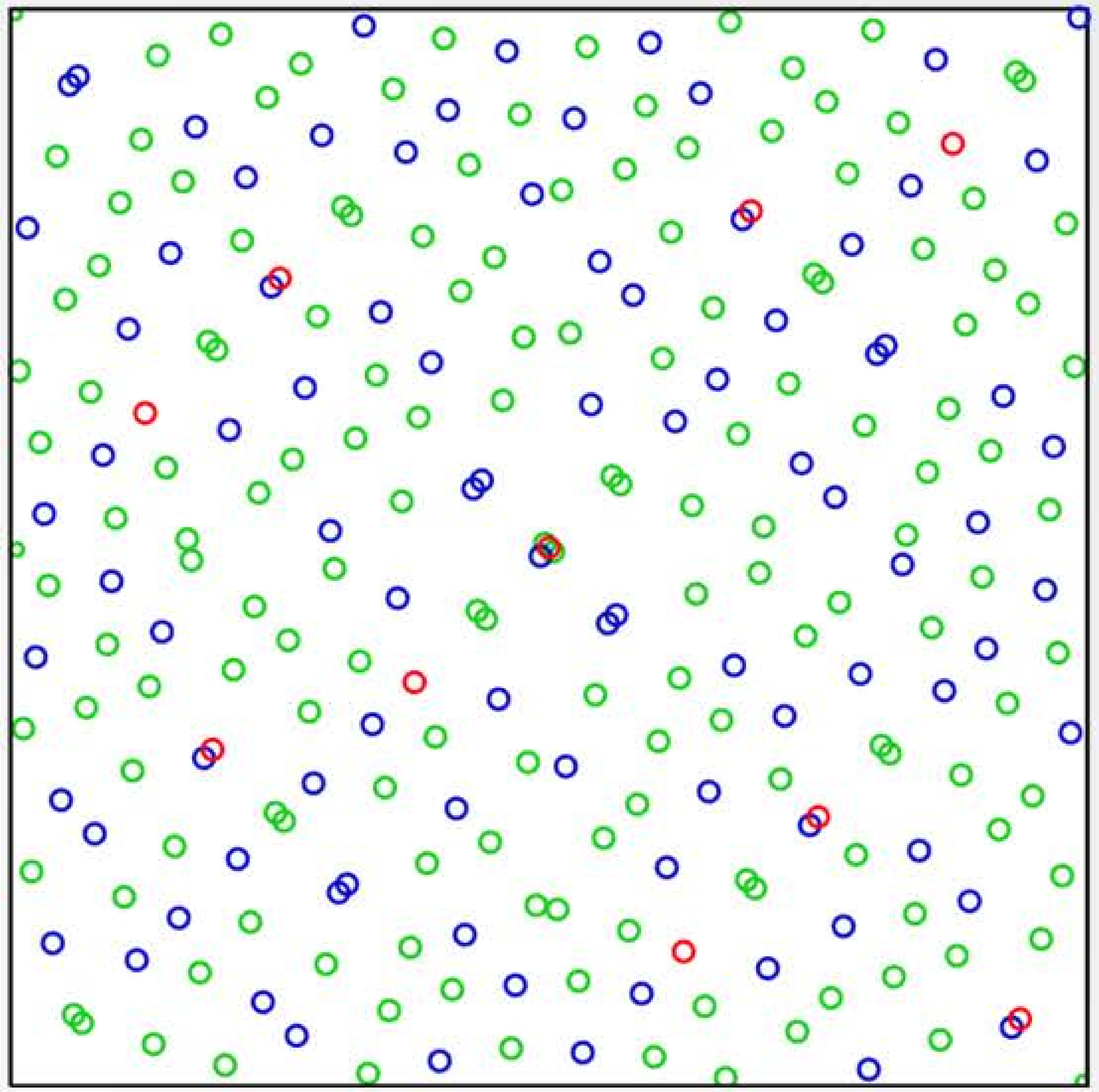
- Non-uniform samples
- Un-equal sample weights



Monte Carlo Integration

Quasi Monte Carlo

- Uniform determinstic samples
- Equal sample weights



Monte Carlo Integration

Uniformity vs. Randomness

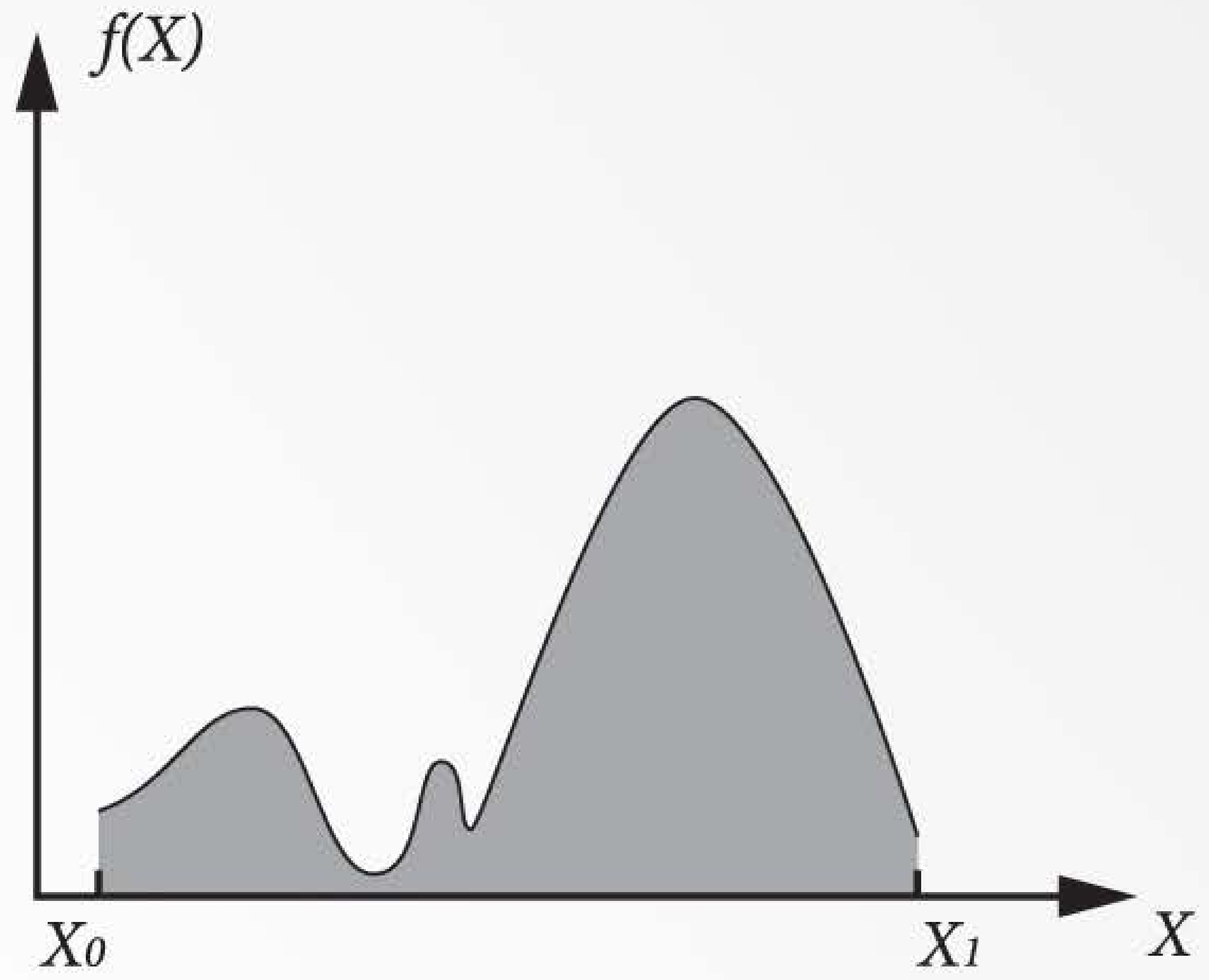
- Distribution hard to craft
- Correlated samples leads to alias

Monte Carlo Integration

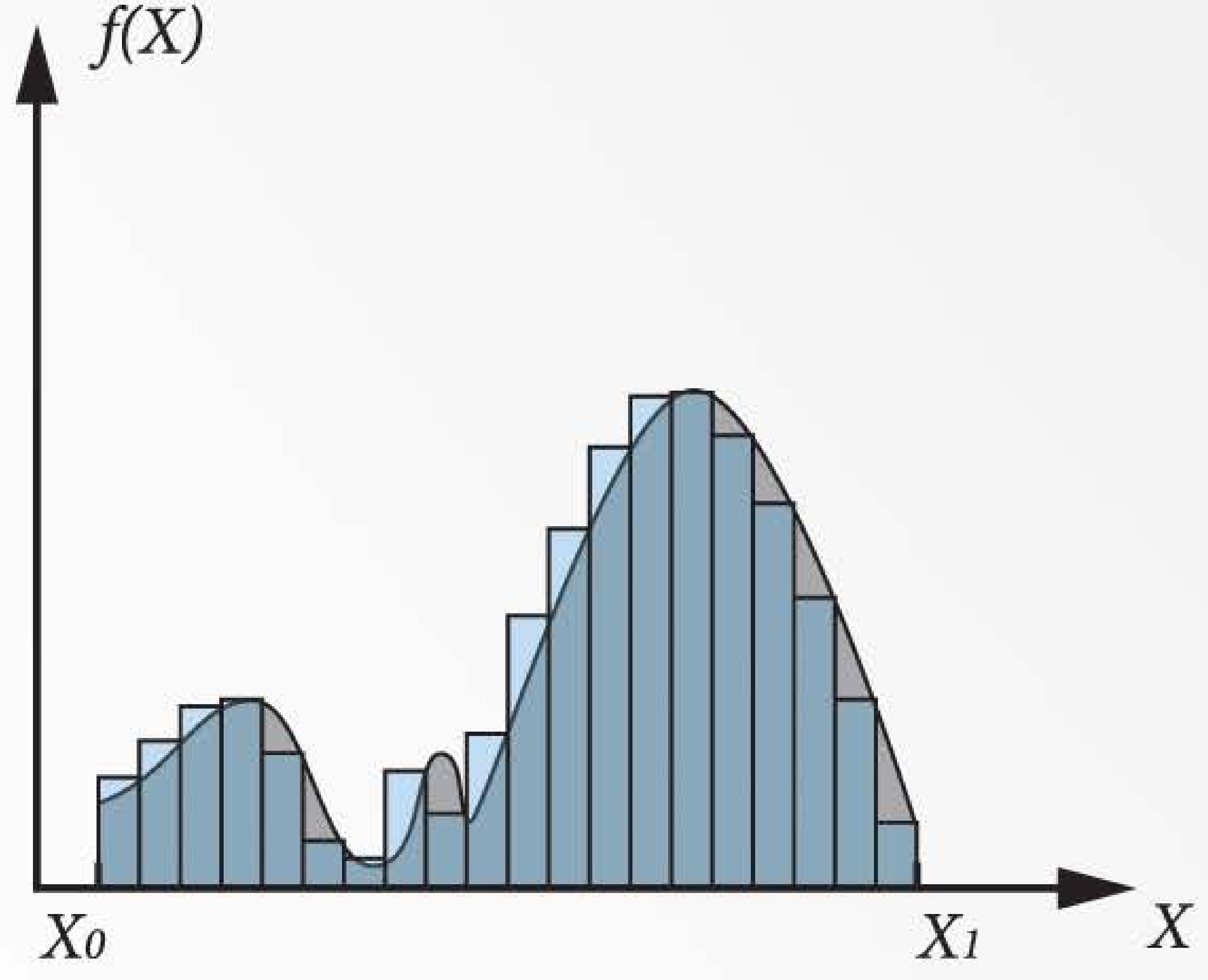
Geometrical Sample Reweighting

- Maintain randomness
- Reweighted uniformity

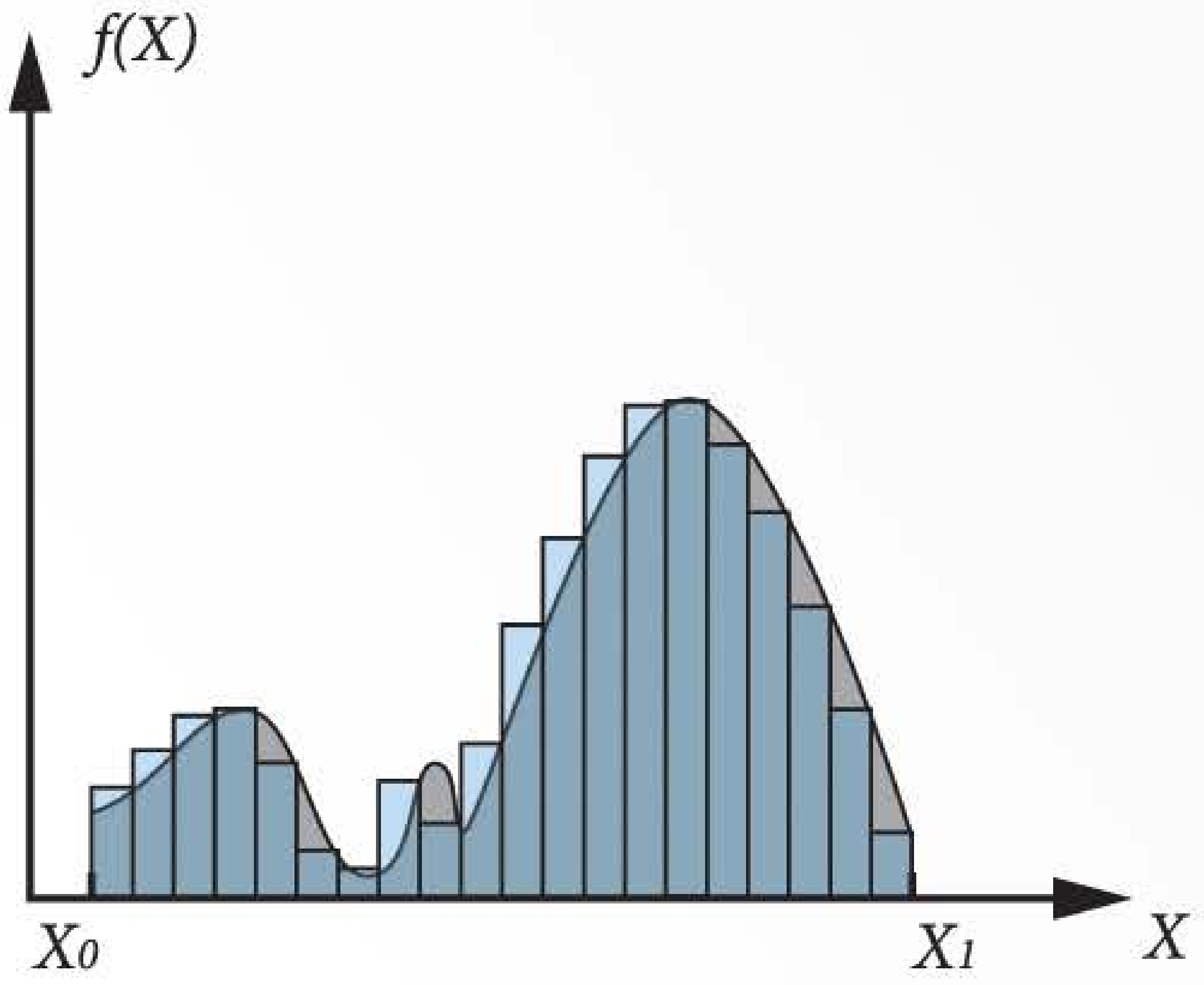
1D Integration with MC



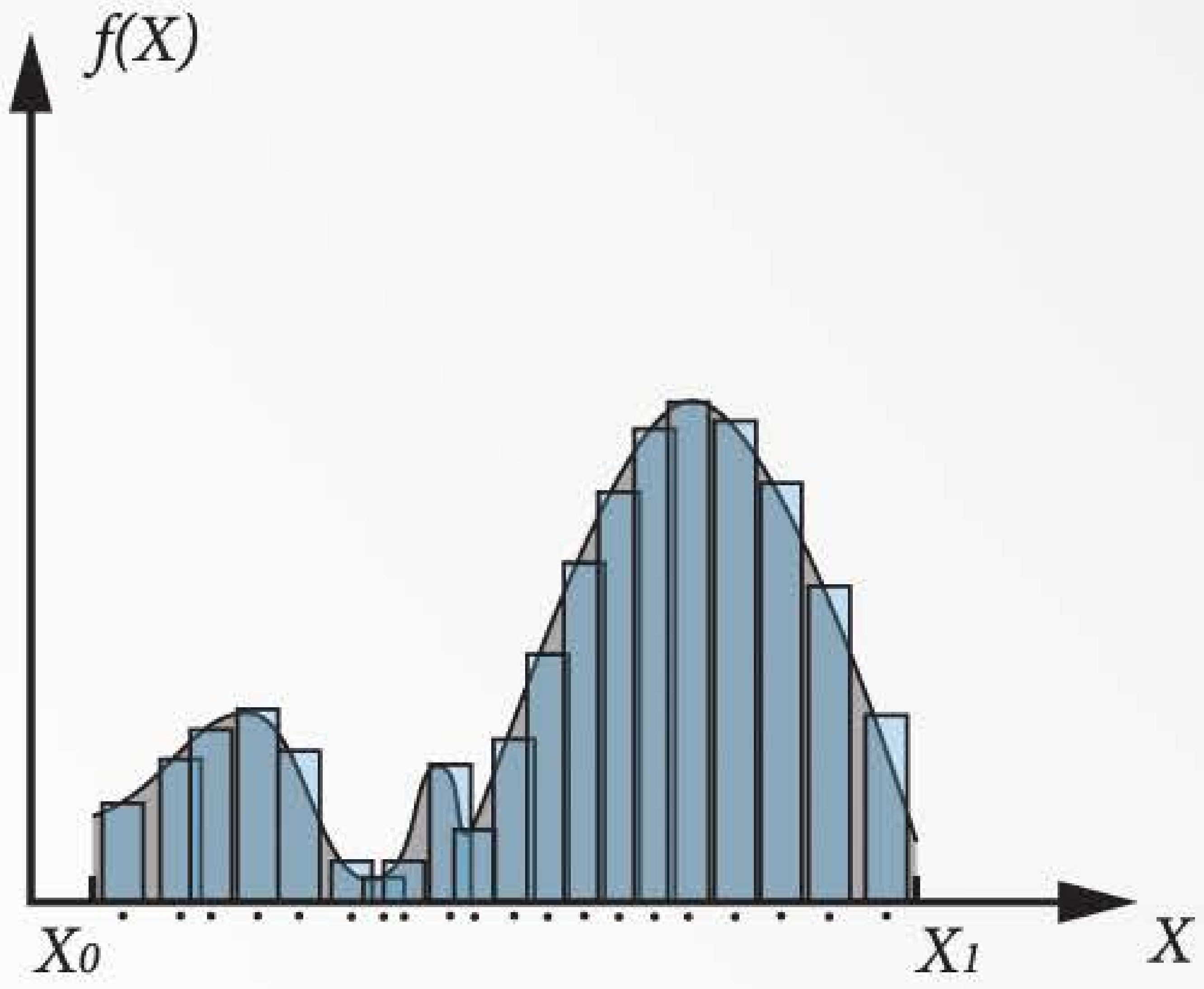
1D Integration with MC



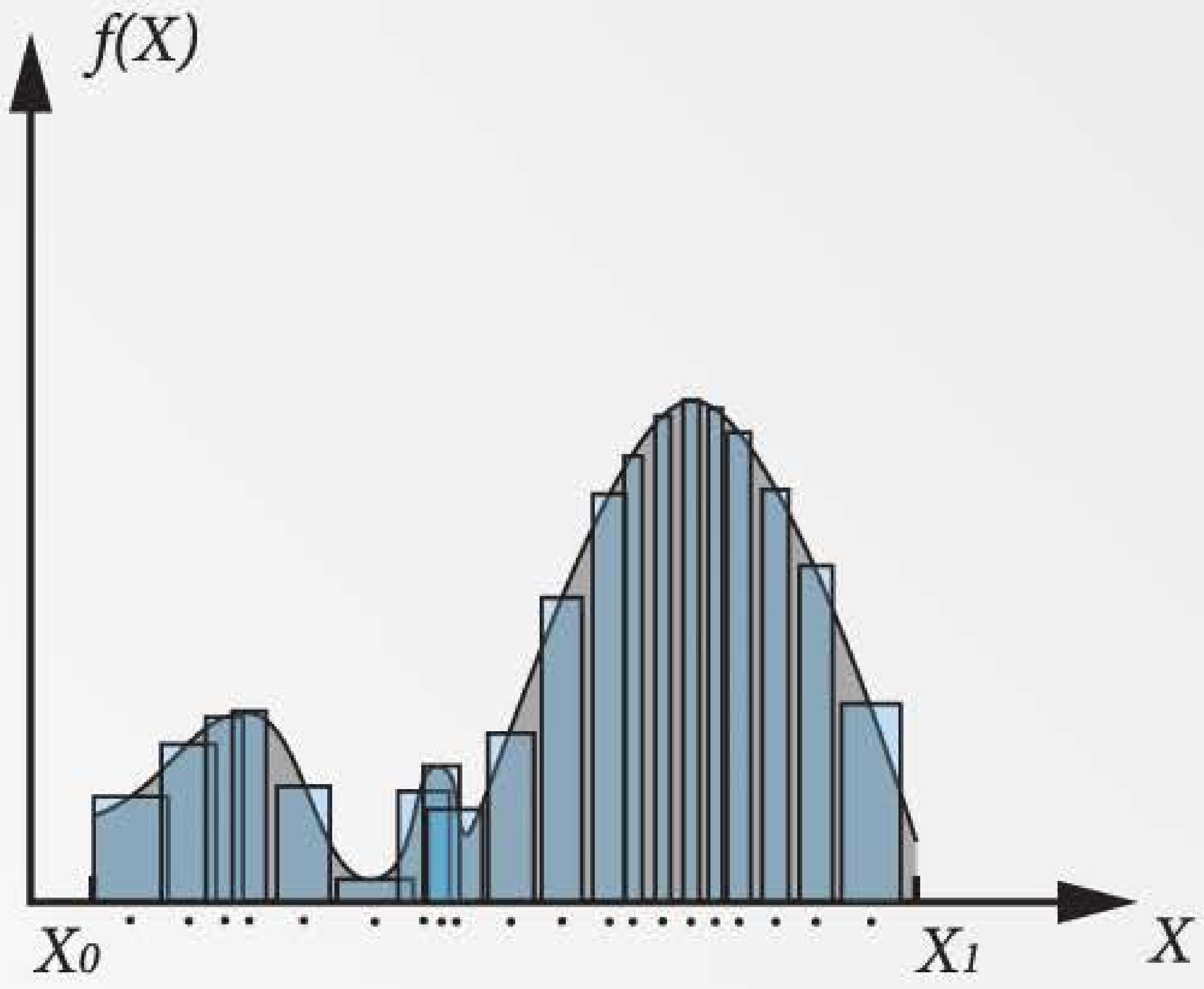
1D Integration with MC



Regular binning

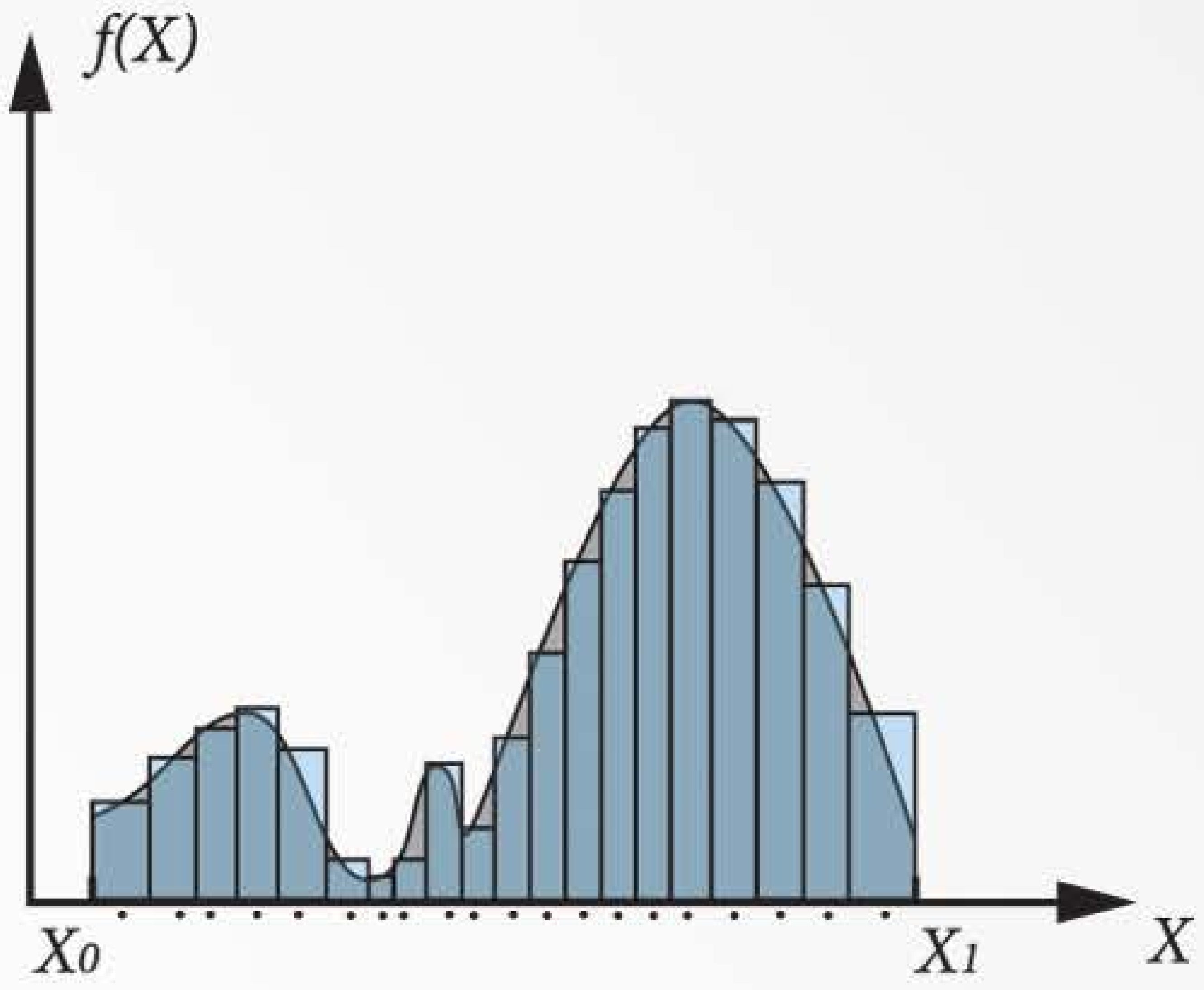


MC w/ random samples



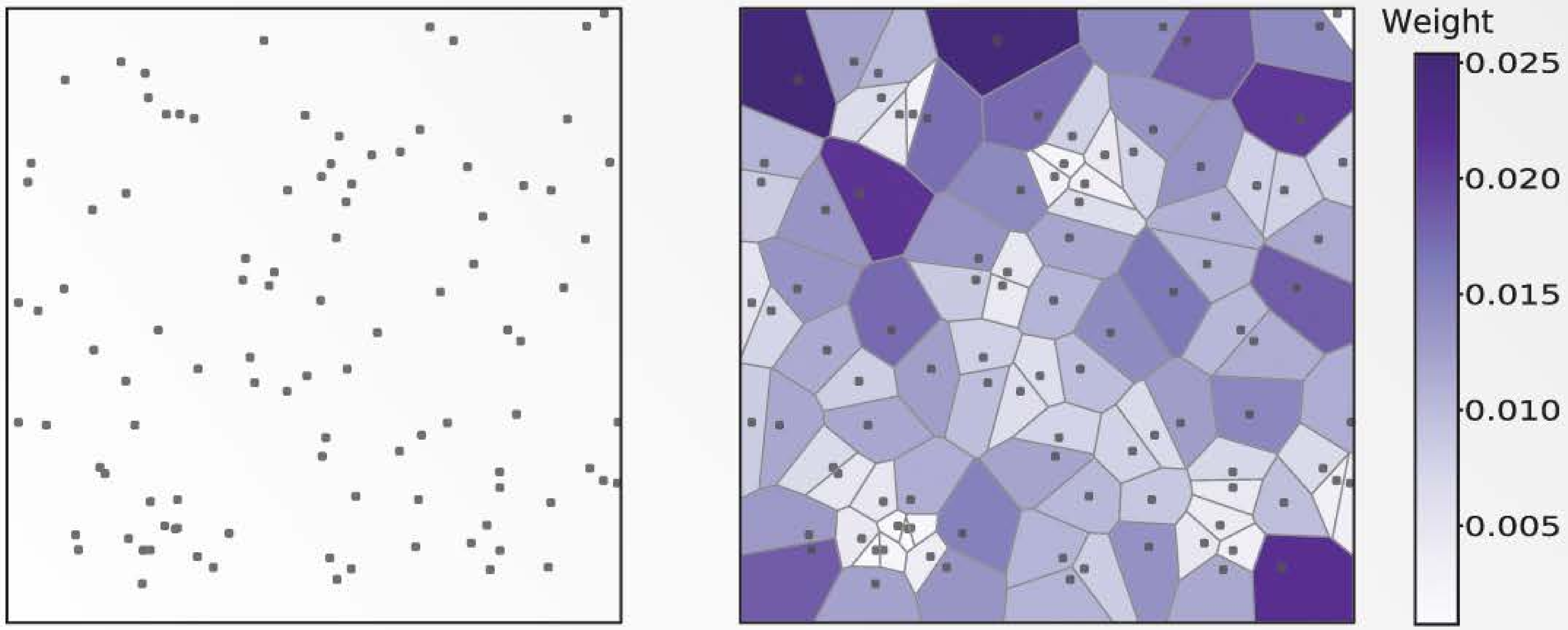
MC w/ importance sampling

1D Integration with MC



MC w/ reweighted
random samples

1D Integration with MC



1D Integration with MC

$$\hat{I}_C = \sum_{i=1}^{N} w_C(x_i) f(x_i), \text{where } w_C(x_i) = \frac{|V_i|}{|\Omega|}$$

1D Integration with MC

$$X = \{x_1, x_2, x_3\}, x_i \in (0, 1), x_1 < x_2 < x_3$$

1D Integration with MC

$$X = \{x_1, x_2, x_3\}, x_i \in (0,1), x_1 < x_2 < x_3$$

1D Integration with MC

$$E[x_1] = \frac{1}{4}, E[x_2] = \frac{2}{4}, E[x_3] = \frac{3}{4}$$

1D Integration with MC

$$E[w_C(x_1)] = \frac{3}{8}, E[w_C(x_2)] = \frac{2}{8}, E[w_C(x_3)] = \frac{3}{8}$$

1D Integration with MC

$$\hat{I}_C = \sum_{i=1}^{N} w_C(x_i) f(x_i), \text{ where } w_C(x_i) = \frac{|V_i|}{|\Omega|}$$

BIASED

Contribution

Unbiased solution to 1D uniform reweighting

Start from definition

$$E[\hat{I}] = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} f(x_i) = \int f(x)\mathrm{d}x$$

Contribution

Unbiased solution to 1D uniform reweighting

Contribution

Unbiased solution to 1D uniform reweighting

- Start from definition
- Prove what we already know

Start from Definition

$$\|\{x\}\| = N$$

$$\mathbb{F}(\{x\}) = \frac{1}{N}\sum_{i=1}^{N}\frac{f(x_i)}{p(x_i)}$$

Start from Definition

$$E\left[\frac{1}{N}\sum_{i=1}^{N}\frac{f(x_i)}{p(x_i)}\right] = E\left[\mathbb{F}(\{x\})\right]$$

Start from Definition

$$\mathbb{P}(\{x\}) = N! \prod_{i=1}^{N} p(x_i)$$

$$p(x_i) = 1$$

Start from Definition

$$E[\mathbb{F}(\{x\})] = \int \mathbb{F}(\{x\})\mathbb{P}(\{x\})\mathrm{d}\{x\} = \int \mathbb{F}(\{x\})\mathrm{d}\{x\}$$

Start from Definition

$$
\begin{aligned}
\mathbb{E}\left[\hat{I}_{MC}\right] &= \int_{(0,1)^N} \mathbb{F}(\{x_i\})P(\{x_i\})\mathrm{d}\{x_i\} \\
&= \int_0^1 \int_0^1 \cdots \int_0^1 \left[\sum_{i=1}^{N} \frac{1}{N}\frac{f(x_i)}{p(x_i)}\right] N! \prod_{i=1}^{N} p(x_i)\mathrm{d}\{x_i\} \\
&= \int_0^1 \int_{x_1}^1 \cdots \int_{x_{N-1}}^1 \left[\sum_{i=1}^{N} \frac{1}{N} f(x_i)\right] N!\mathrm{d}\{x_i\} \\
&= N! \int_0^1 \int_{x_1}^1 \cdots \int_{x_{N-1}}^1 \\
&\left[\frac{1}{N} f(x_1) + \frac{1}{N} f(x_2) + \cdots + \frac{1}{N} f(x_N)\right] \mathrm{d}x_N \mathrm{d}x_{N-1} \cdots \mathrm{d}x_1
\end{aligned}
$$

Start from Definition

$$\int_0^1 \int_{x_1}^1 \cdots \int_{x_{N-1}}^1 \mathrm{d}x_N \mathrm{d}x_{N-1} \cdots \mathrm{d}x_1 = \int_0^1 \int_0^{x_N} \cdots \int_0^{x_2} \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_N$$

Start from Definition

$$\int_0^{x_{i+1}} \int_0^{x_i} \cdots \int_0^{x_2} \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_i = \frac{x_{i+1}^i}{i!},$$

$$\int_{x_{i-1}}^1 \int_{x_i}^1 \cdots \int_{x_{N-1}}^1 \mathrm{d}x_N \mathrm{d}x_{N-1} \cdots \mathrm{d}x_i = \frac{(1-x_{i-1})^{N-i+1}}{(N-i+1)!}.$$

Start from Definition

$$\begin{aligned}\mathbb{E}\left[\hat{I}_{MC}\right] &= N!\int_0^1\int_{x_1}^1\cdots\int_{x_{N-1}}^1 \\ &\quad \left[\frac{1}{N}f(x_1)+\frac{1}{N}f(x_2)+\cdots+\frac{1}{N}f(x_N)\right]\mathrm{d}x_N\mathrm{d}x_{N-1}\cdots\mathrm{d}x_1 \\ &= N!\cdot\frac{1}{N}\left[\int_0^1 f(x_1)\int_{x_1}^1\cdots\int_{x_{N-1}}^1 \mathrm{d}x_N\cdots\mathrm{d}x_2\mathrm{d}x_1+\cdots\right. \\ &\quad +\int_0^1\cdots\int_{x_{i-1}}^1 f(x_i)\cdots\int_{x_{N-1}}^1 \mathrm{d}x_N\mathrm{d}x_{N-1}\cdots\mathrm{d}x_1+\cdots \\ &\quad \left.+\int_0^1\int_{x_1}^1\cdots\int_{x_{N-1}}^1 f(x_N)\mathrm{d}x_N\mathrm{d}x_{N-1}\cdots\mathrm{d}x_1\right]\end{aligned}$$

Start from Definition

$$
\begin{aligned}
&= N! \cdot \frac{1}{N} \left[\int_0^1 \frac{(1-x_1)^{N-1}}{(N-1)!} f(x_1) dx_1 + \cdots \right. \\
&\quad + \int_{x_{i-1}}^1 \frac{x_i^{i-1}}{(i-1)!} \frac{(1-x_i)^{N-i}}{(N-i)!} f(x_i) \mathrm{d}x_i + \cdots \\
&\quad \left. + \int_{x_{N-1}}^1 \frac{x_N^{N-1}}{(N-1)!} f(x_N) \mathrm{d}x_N \right]
\end{aligned}
$$

Start from Definition

$$= N! \cdot \frac{1}{N} \left[\int_0^1 \frac{(1-x)^{N-1}}{(N-1)!} f(x) \mathrm{d}x + \cdots + \right.$$

$$\int_0^1 \frac{x^{i-1}}{(i-1)!} \frac{(1-x)^{N-i}}{(N-i)!} f(x) \mathrm{d}x + \cdots$$

$$\left. + \int_0^1 \frac{x^{N-1}}{(N-1)!} f(x) \mathrm{d}x \right]$$

Start from Definition

$$= N! \cdot \frac{1}{N} \int_0^1 \sum_{i=1}^{N} \frac{x^{i-1}}{(i-1)!} \frac{(1-x)^{N-i}}{(N-i)!} f(x) \mathrm{d}x$$

$$= \int_0^1 \sum_{i=1}^{N} \frac{(N-1)!}{(i-1)!(N-i)!} x^{i-1} (1-x)^{N-i} f(x) \mathrm{d}x$$

Start from Definition

$$= \int_0^1 \cdot [x + (1-x)]^{N-1} \cdot f(x)\mathrm{d}x = \int_0^1 f(x)\mathrm{d}x.$$

Integrating the Consistent Reweighting Estimator

Repeating the same process

Integrating the Consistent Reweighting Estimator

$$\mathbb{E}\left[\hat{I}_C\right] =$$

$$N! \int_0^1 \int_{x_1}^1 \cdots \int_{x_{N-1}}^1$$

$$\left[\frac{x_1 + x_2}{2} f(x_1) + \sum_{i=2}^{N-1} \frac{x_{i+1} - x_{i-1}}{2} f(x_i) + \right.$$

$$\left.\left(1 - \frac{x_{N-1} + x_N}{2}\right) f(x_N)\right] \mathrm{d}x_N \mathrm{d}x_{N-1} \cdots \mathrm{d}x_1$$

Integrating the Consistent Reweighting Estimator

$$= N! \cdot \int_0^1 \left\{ \frac{1}{2} \cdot \frac{x \cdot (1-x)^{N-1}}{(N-1)!} + \frac{1}{2} \cdot \frac{(1-x)^{N-1}\left[(N-1)\cdot x + 1\right]}{N!} \right.$$

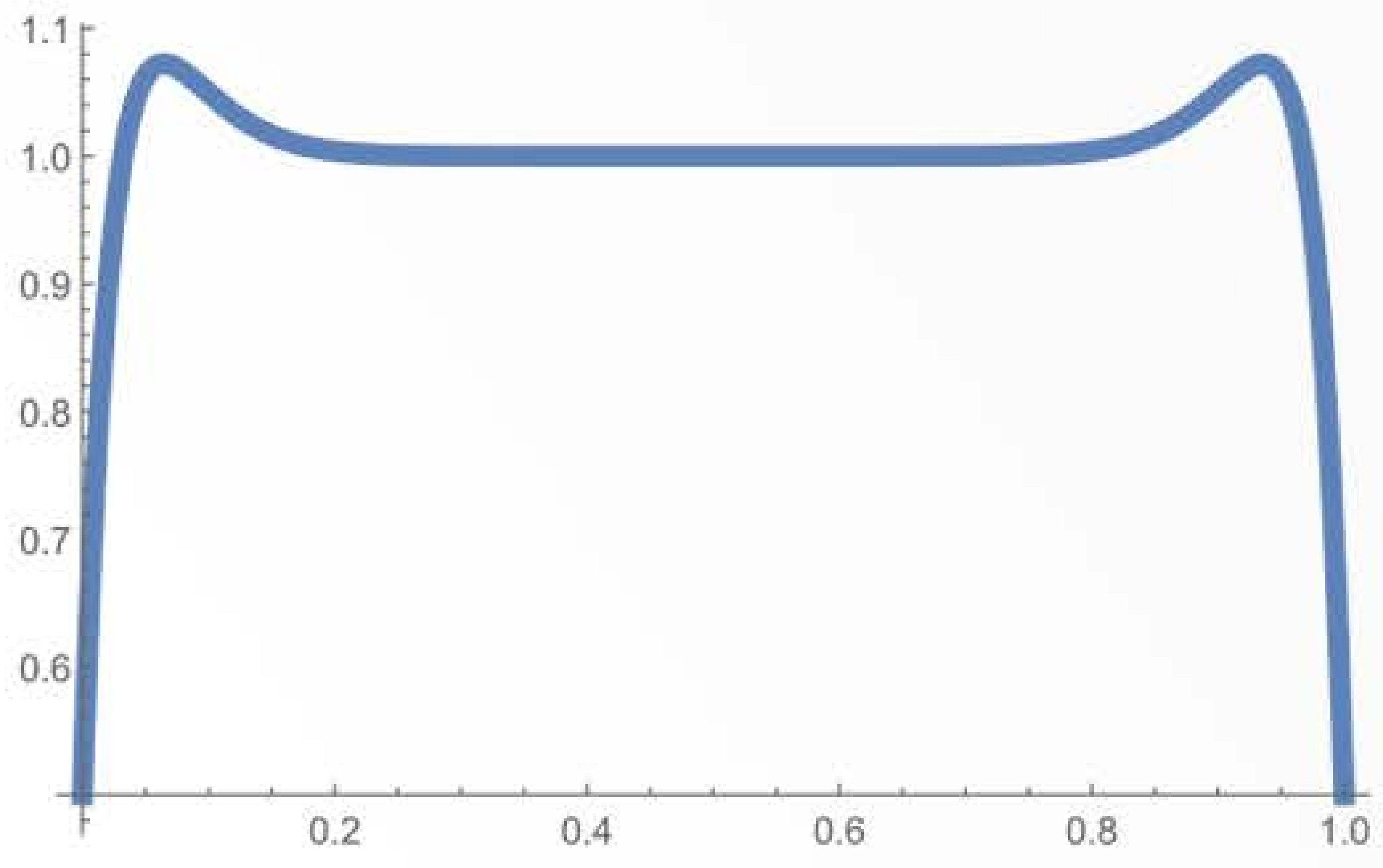
$$+ \sum_{i=2}^{N-1} \left[\frac{1}{2} \cdot \frac{(1-x)^{N-i}\left[(N-i)\cdot x+1\right]\cdot x^{i-1}}{(N-i+1)!(i-1)!} - \frac{1}{2} \cdot \frac{(1-x)^{N-i}\cdot x^{i}}{i(N-i)!(i-2)!} \right]$$

$$\left. + \left[\frac{x^{N-1}}{(N-1)!} - \frac{1}{2} \cdot \frac{x^N}{N(N-2)!} - \frac{1}{2} \cdot \frac{x^N}{(N-1)!} \right] \right\} \cdot f(x)\mathrm{d}x$$

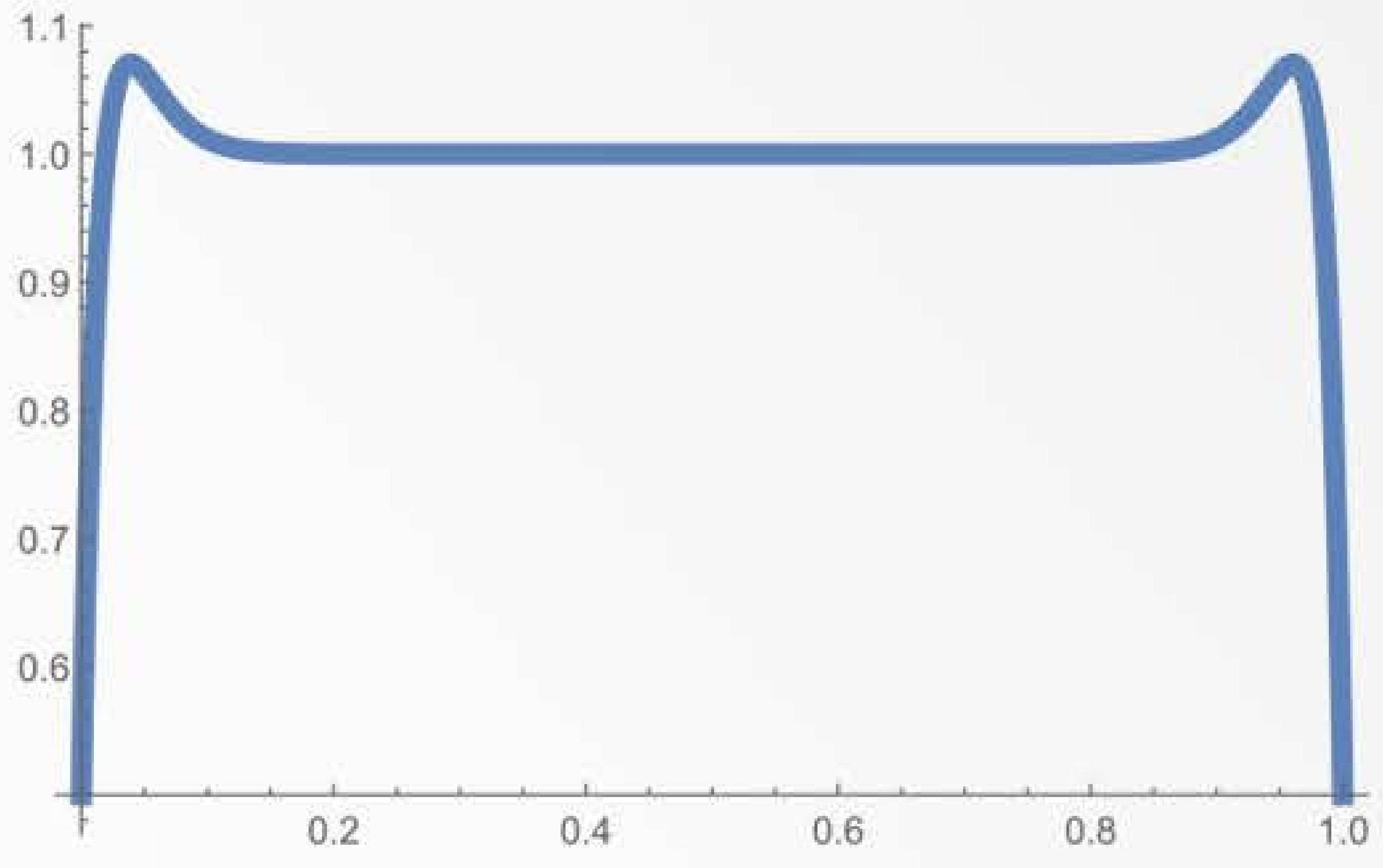
Integrating the Consistent Reweighting Estimator

$$= \int_0^1 \frac{(N - Nx - x)x^{N-1} + (Nx + x - 1)(1 - x)^{N-1} + 2}{2} \cdot f(x)\mathrm{d}x$$

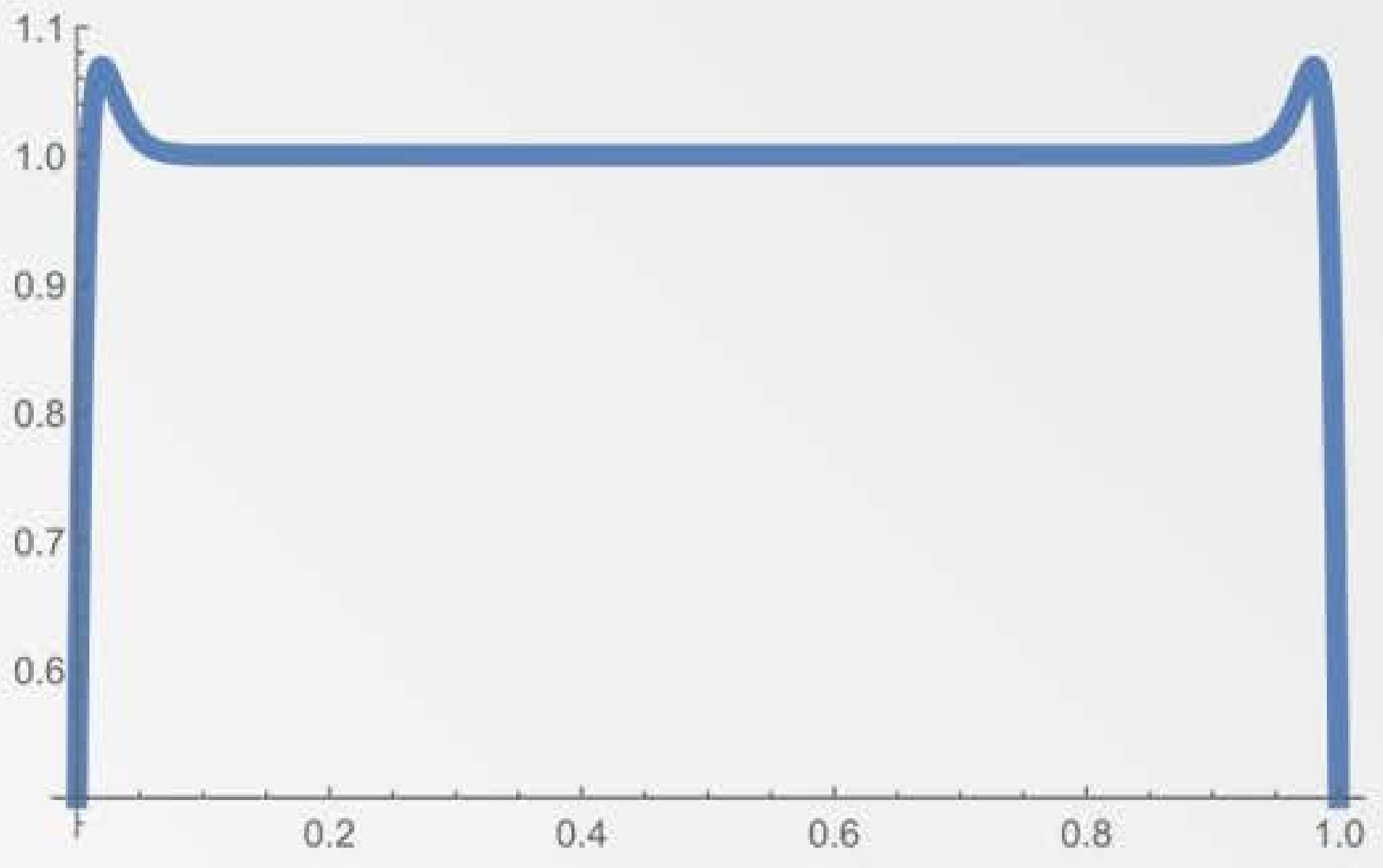
Integrating the Consistent Reweighting Estimator



N=30



N=50



N=100

Integrating the Consistent Reweighting Estimator

$$g(x) = \frac{(N - Nx - x)x^{N-1} + (Nx + x - 1)(1 - x)^{N-1} + 2}{2}$$

Integrating the Consistent Reweighting Estimator

$$E[\mathbb{F}(\{x\})] = \int g(x) f(x) \mathrm{d}x$$

Deriving the Unbiased Reweighting Estimator

$$\mathbb{E}\left[\hat{I}_C^F\right] = \int_0^1 g(x) \cdot F(x)\mathrm{d}x = \int_0^1 g(x) \cdot \frac{f(x)}{g(x)}\mathrm{d}x$$

$$= \int_0^1 f(x)\mathrm{d}x = \mathbb{E}\left[\hat{I}\right].$$

Deriving the Unbiased Reweighting Estimator

$$\hat{I}_{GR} = \sum_{i=1}^{N} w_{GR}(x_i) f(x_i) = \sum_{i=1}^{N} \frac{|V_i|}{|\Omega|} \frac{1}{g(x_i)} f(x_i)$$

Numerical Performance

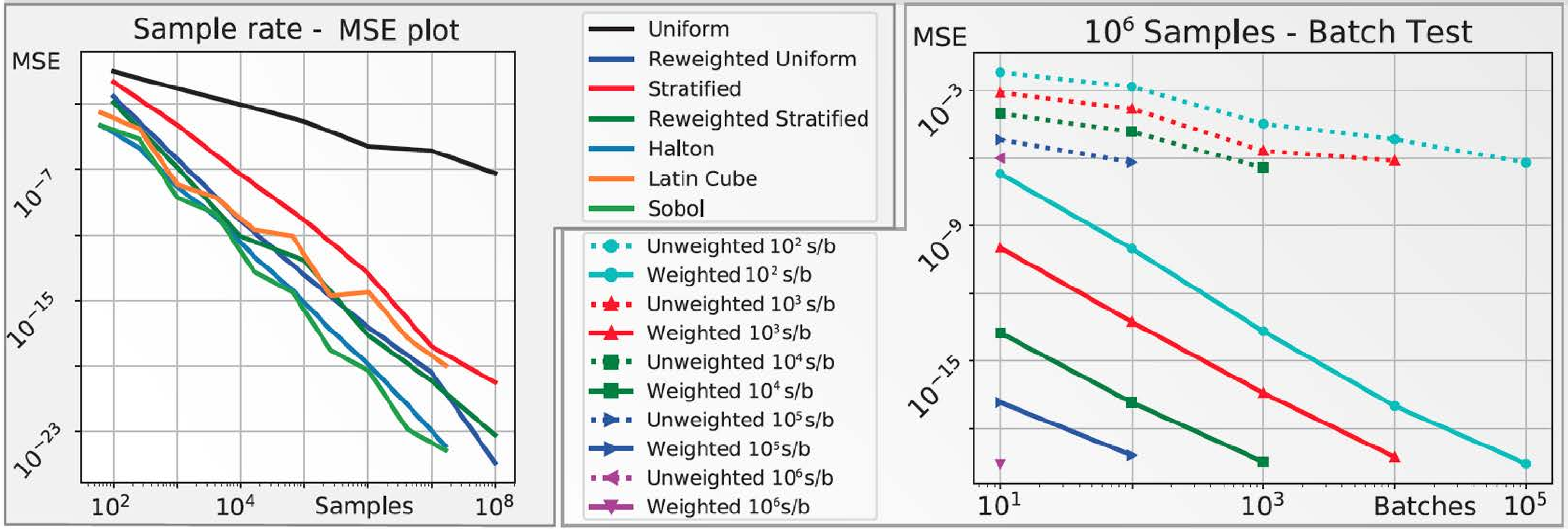
1D Unbiased & 2D Consistent

Numerical Performance

$$f(x) = 10 \times \begin{cases} \sqrt{-x^2 + 0.5x} & x \leq 0.25 \\ -\sqrt{-x^2 + x - 0.1875} + 0.25 & 0.25 < x \leq 0.5 \\ 20 \times (x - 0.5) & 0.5 < x \leq 0.55 \\ 1.0 & 0.55 < x \leq 0.65 \\ -20 \times (x - 0.7) & 0.65 < x \leq 0.7 \\ 0.1 \times \sin(10\pi \cdot (x - 0.7)) & 0.7 < x \leq 0.8 \\ 0.25 \times \sin(10\pi \cdot (x - 0.8)) & 0.8 < x \leq 0.9 \\ 0.5 \times \sin(10\pi \cdot (x - 0.9)) & 0.9 < x \end{cases}$$

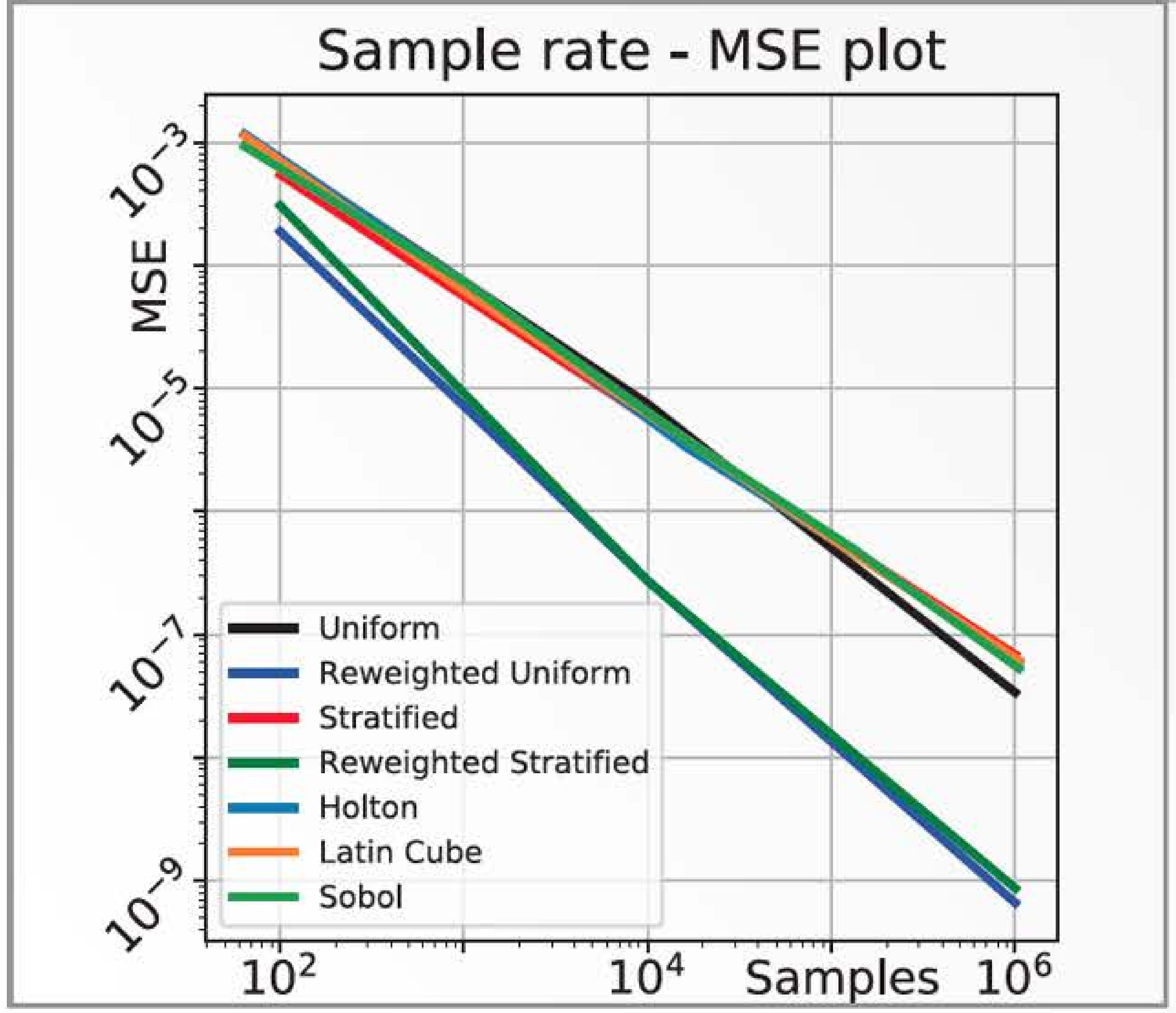
Numerical Performance

1D Sampling



Numerical Performance

2D Sampling



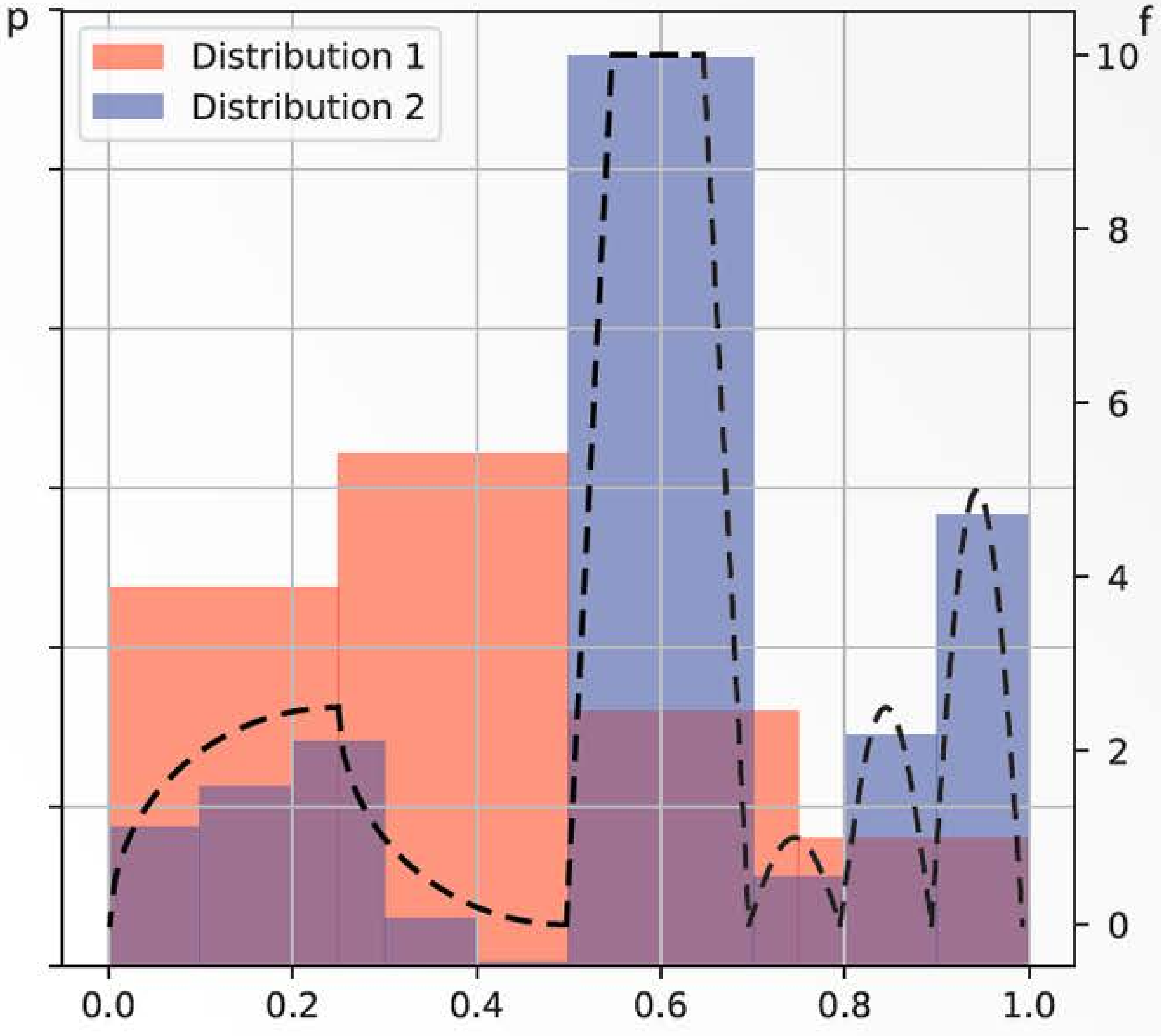
Numerical Performance

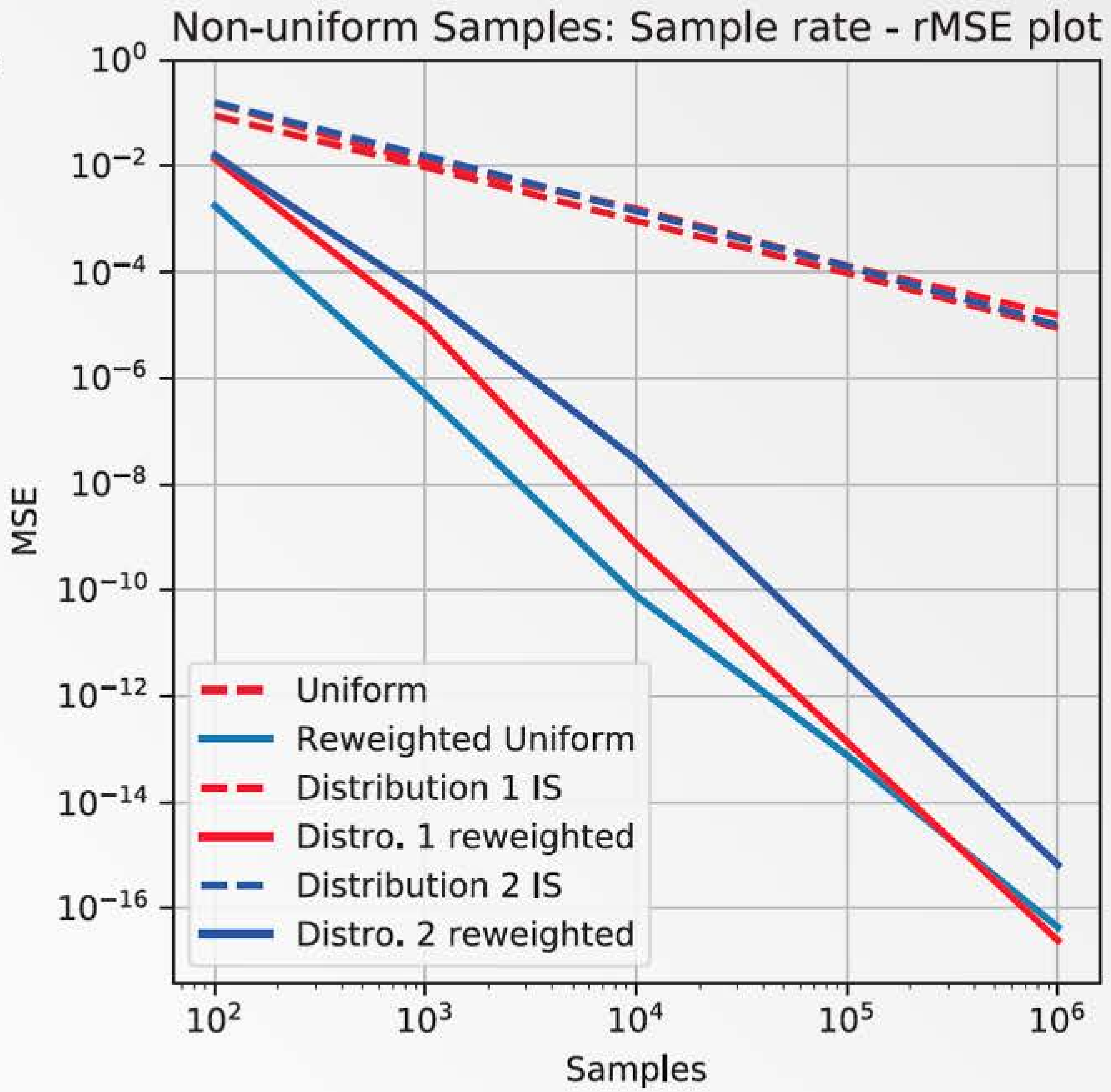
Unbiased solution for uniform samples

Unbiased solution for piece-wise uniform samples

- Commonly used in computer graphics
- Essentially a collection of uniform sample sets

Numerical Performance



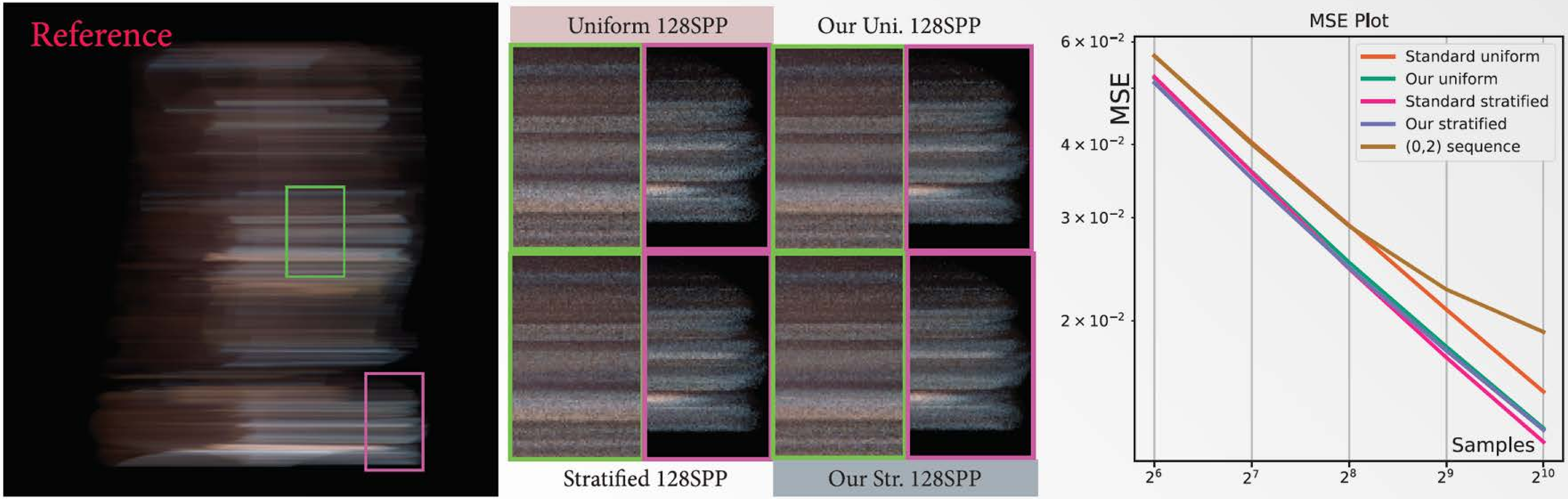


Applying to Rendering

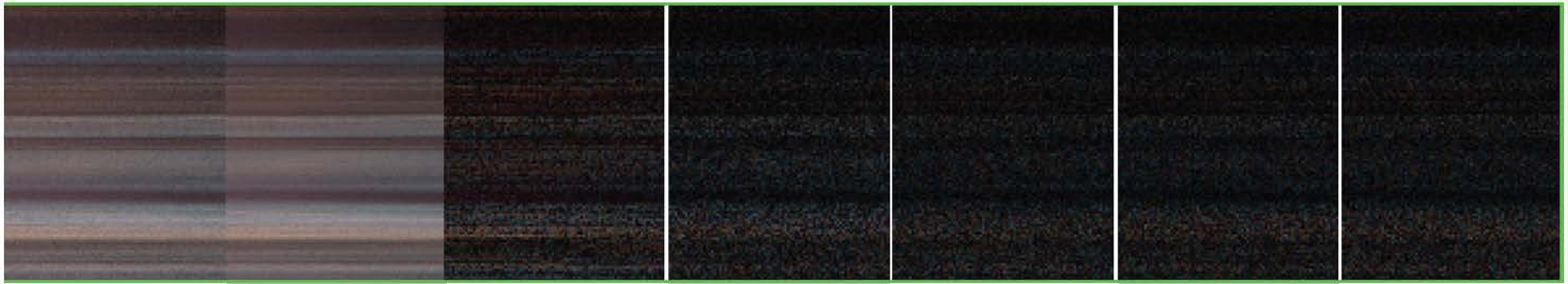
Monte Carlo rendering problems

- 1D temporal sampling
- 1D spectral sampling
- 2D lens sampling
- 2D direct illumination sampling

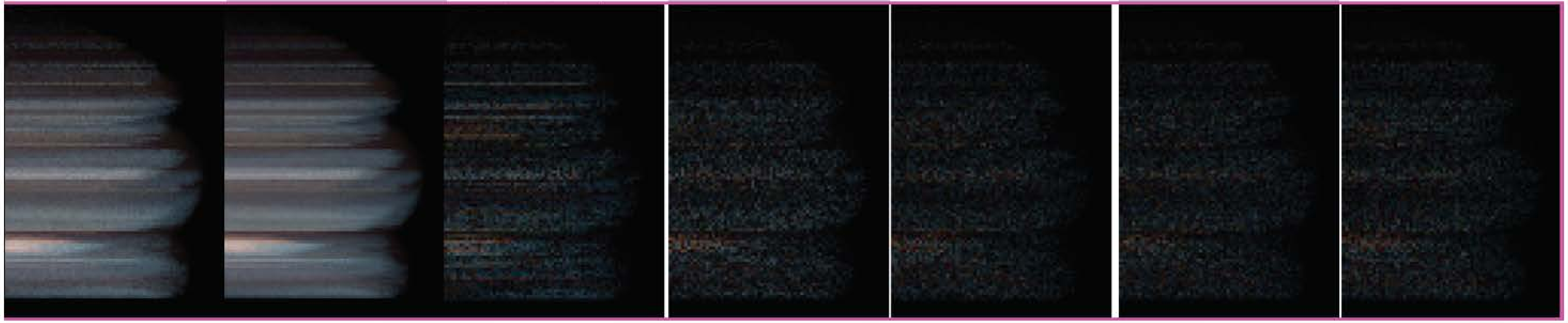
Results


Reference
Uniform 128SPP
Our Uni. 128SPP
Stratified 128SPP
Our Str. 128SPP
MSE Plot
MSE
6×10^{-2}
4×10^{-2}
3×10^{-2}
2×10^{-2}
Standard uniform
Our uniform
Standard stratified
Our stratified
(0,2) sequence
Samples
2^6
2^7
2^8
2^9
2^{10}

Results



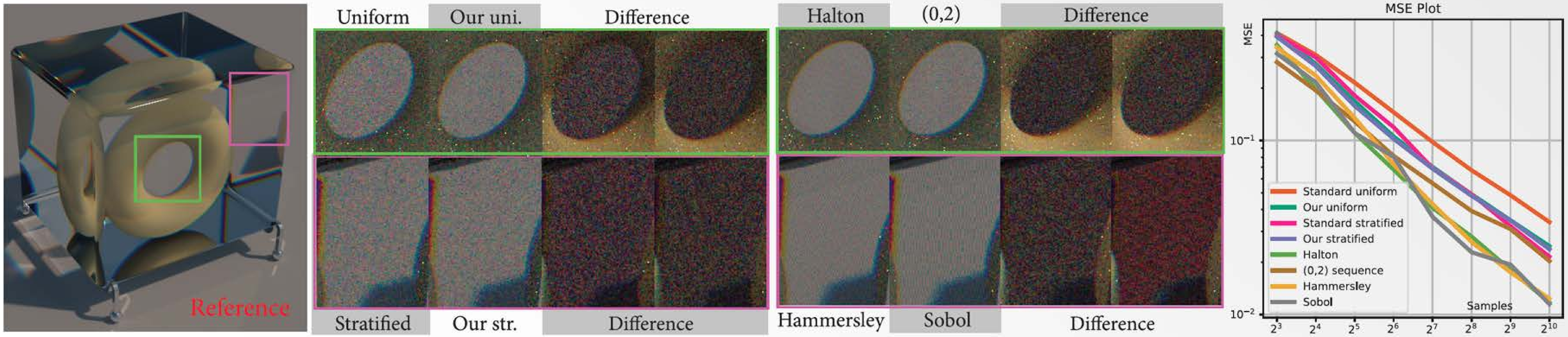
(0,2) seq. 512 SPP	(0,2) seq. 1024 SPP	Diff. 1K SPP (0,2) seq.	Diff. 1K SPP Uniform	Diff. 1K SPP Our uniform	Diff. 1K SPP Stratified	Diff. 1K SPP Our Str.



Results

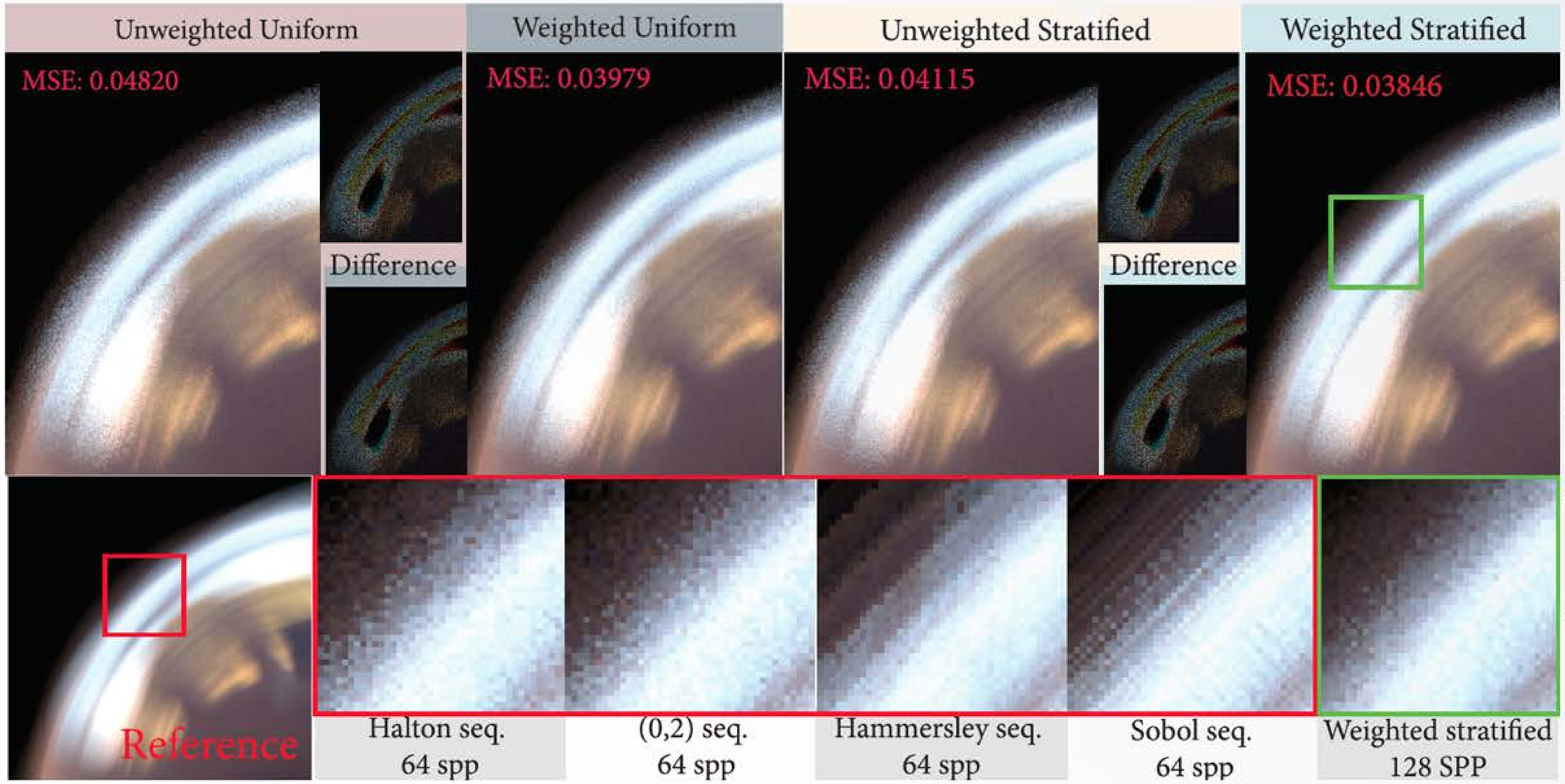

Reference
Uniform
Our uni.
Stratified
Our Str.
Halton
Ham.
(0,2) seq.
Sobol
32 SPP
Diff. 32 SPP
MSE Plot
MSE
10^{-1}
Standard uniform
Our uniform
Standard stratified
Our stratified
Halton
(0,2) sequence
Hammersley
Sobol
Samples
2^3
2^4
2^5
2^6
2^7
2^8
2^9
2^{10}

Results

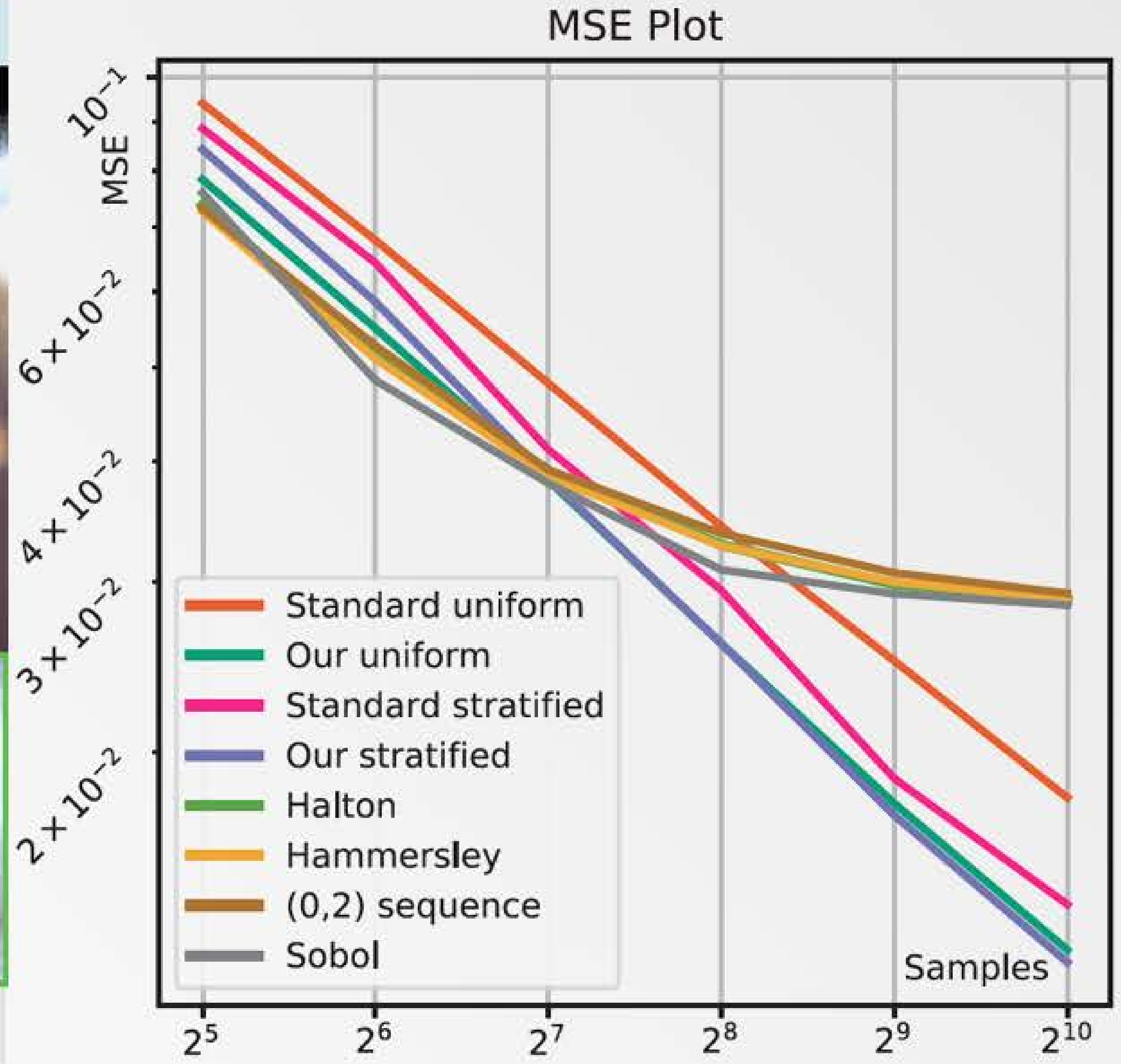


Reference
Uniform
Our uni.
Difference
Stratified
Our str.
Difference
Halton
(0,2)
Difference
Hammersley
Sobol
Difference
MSE Plot
MSE
Samples
Standard uniform
Our uniform
Standard stratified
Our stratified
Halton
(0,2) sequence
Hammersley
Sobol

Results

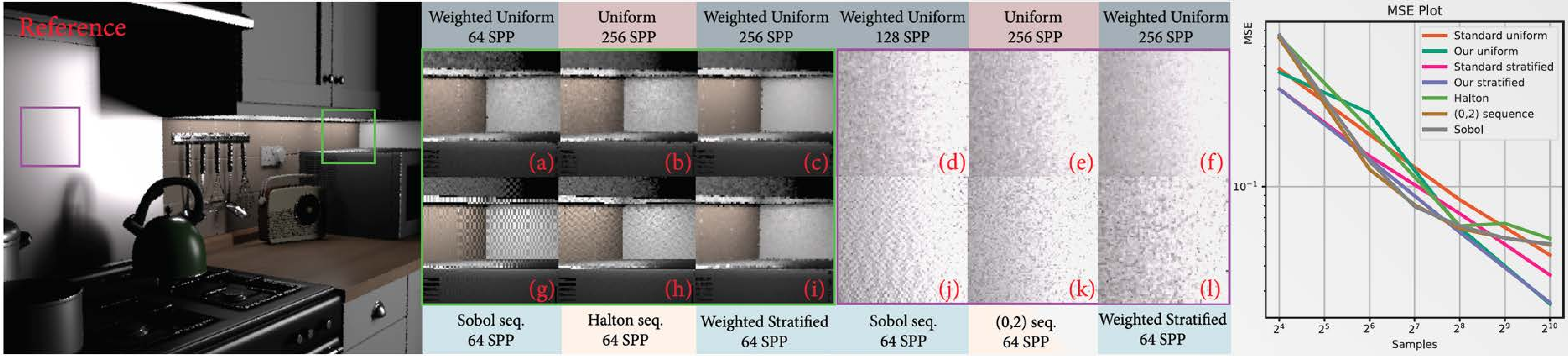


Unweighted Uniform
Weighted Uniform
Unweighted Stratified
Weighted Stratified
MSE: 0.04820
MSE: 0.03979
MSE: 0.04115
MSE: 0.03846
Difference
Difference
Reference
Halton seq.
64 spp
(0,2) seq.
64 spp
Hammersley seq.
64 spp
Sobol seq.
64 spp
Weighted stratified
128 SPP

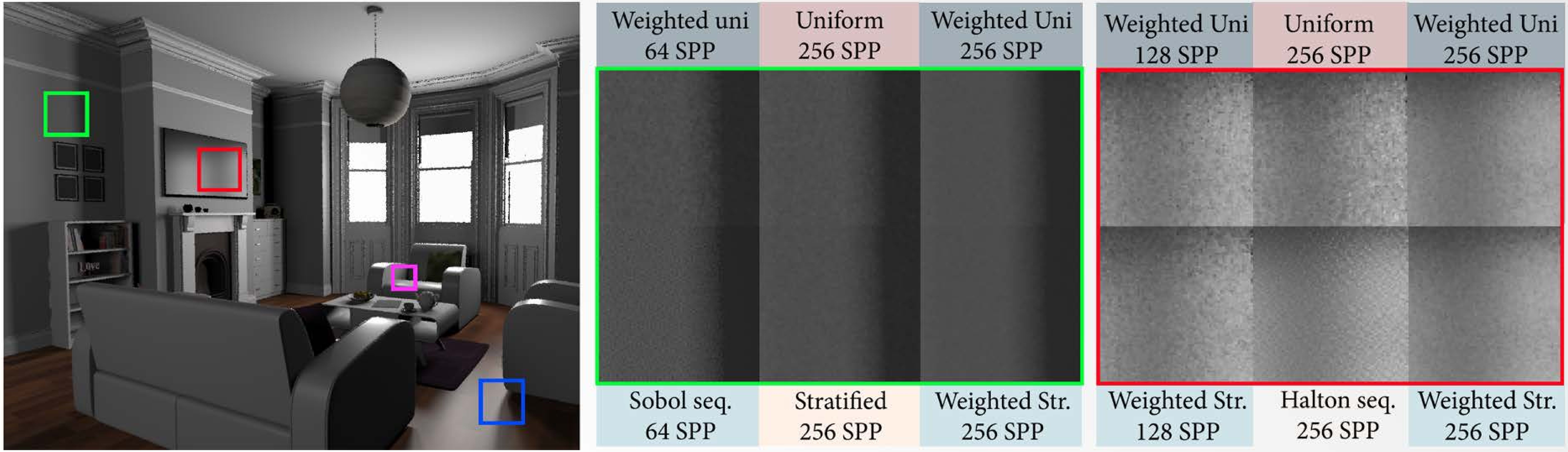


MSE Plot
MSE
10^{-1}
6×10^{-2}
4×10^{-2}
3×10^{-2}
2×10^{-2}
Standard uniform
Our uniform
Standard stratified
Our stratified
Halton
Hammersley
(0,2) sequence
Sobol
Samples
2^5
2^6
2^7
2^8
2^9
2^{10}

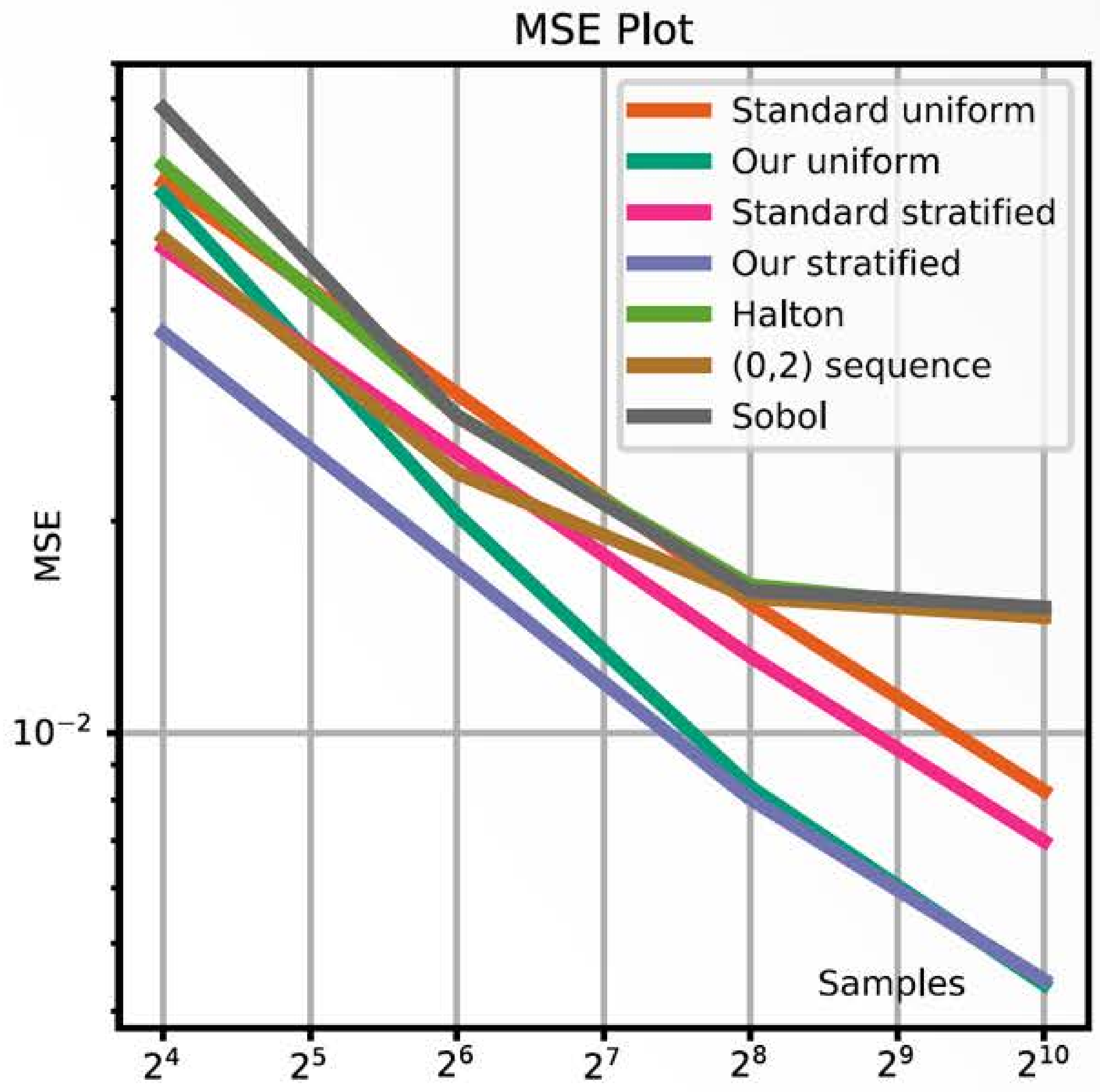
Results

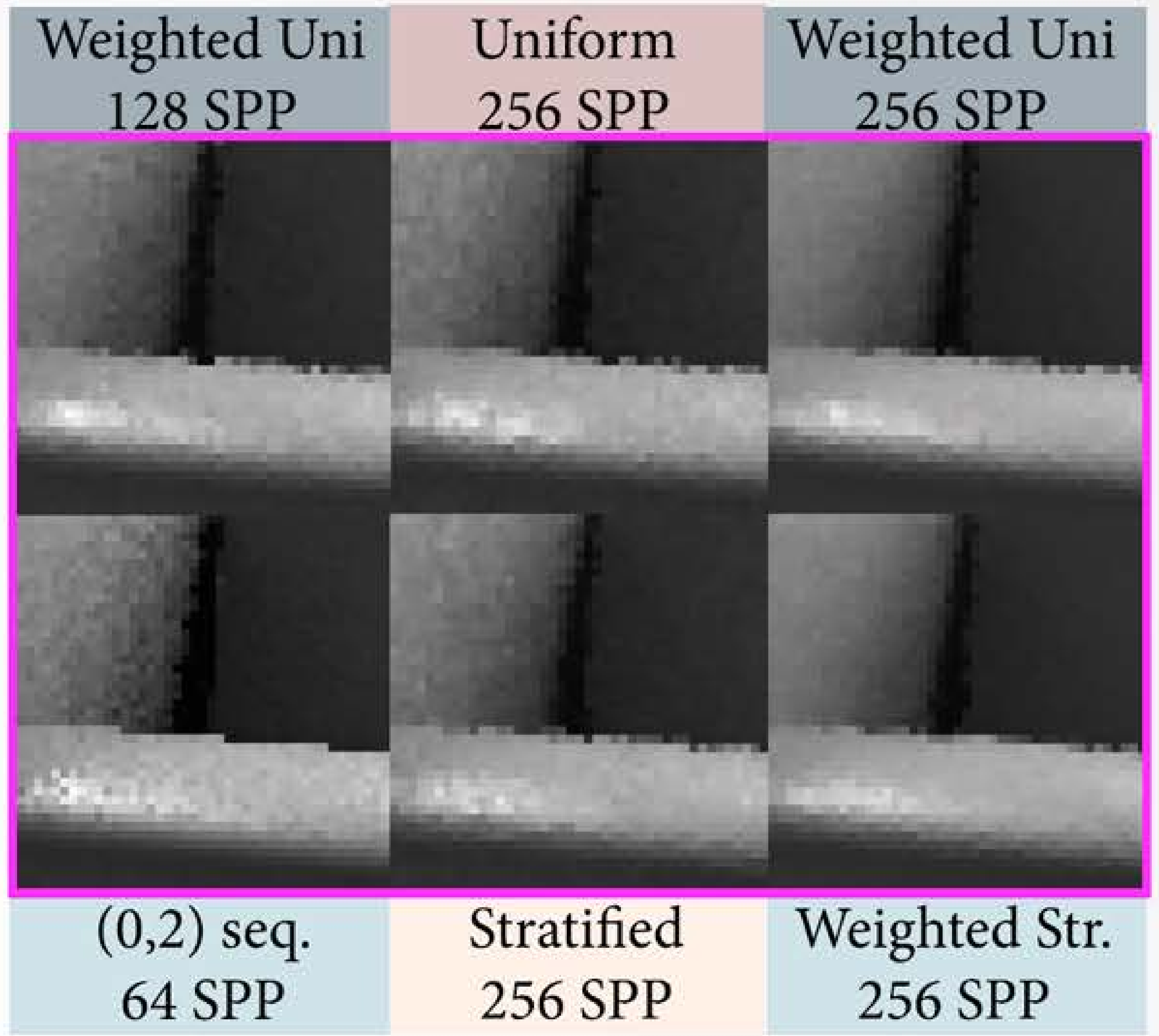

Reference
Weighted Uniform 64 SPP
Uniform 256 SPP
Weighted Uniform 256 SPP
Weighted Uniform 128 SPP
Uniform 256 SPP
Weighted Uniform 256 SPP
(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)
(j)
(k)
(l)
Sobol seq. 64 SPP
Halton seq. 64 SPP
Weighted Stratified 64 SPP
Sobol seq. 64 SPP
(0,2) seq. 64 SPP
Weighted Stratified 64 SPP
MSE Plot
MSE
Standard uniform
Our uniform
Standard stratified
Our stratified
Halton
(0,2) sequence
Sobol
10^{-1}
2^4
2^5
2^6
2^7
2^8
2^9
2^{10}
Samples

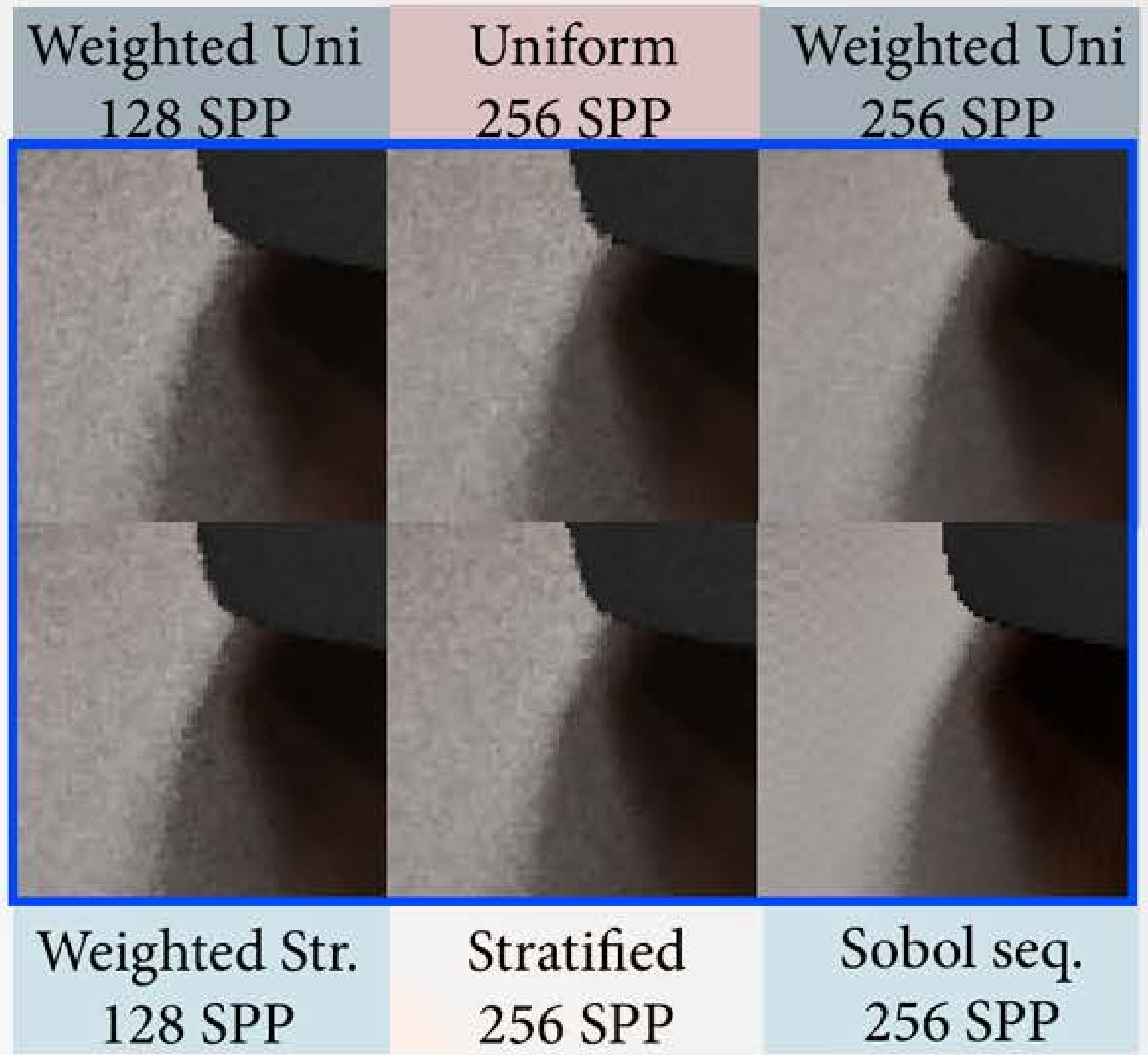
Results


Weighted uni
64 SPP
Uniform
256 SPP
Weighted Uni
256 SPP
Weighted Uni
128 SPP
Uniform
256 SPP
Weighted Uni
256 SPP
Sobol seq.
64 SPP
Stratified
256 SPP
Weighted Str.
256 SPP
Weighted Str.
128 SPP
Halton seq.
256 SPP
Weighted Str.
256 SPP

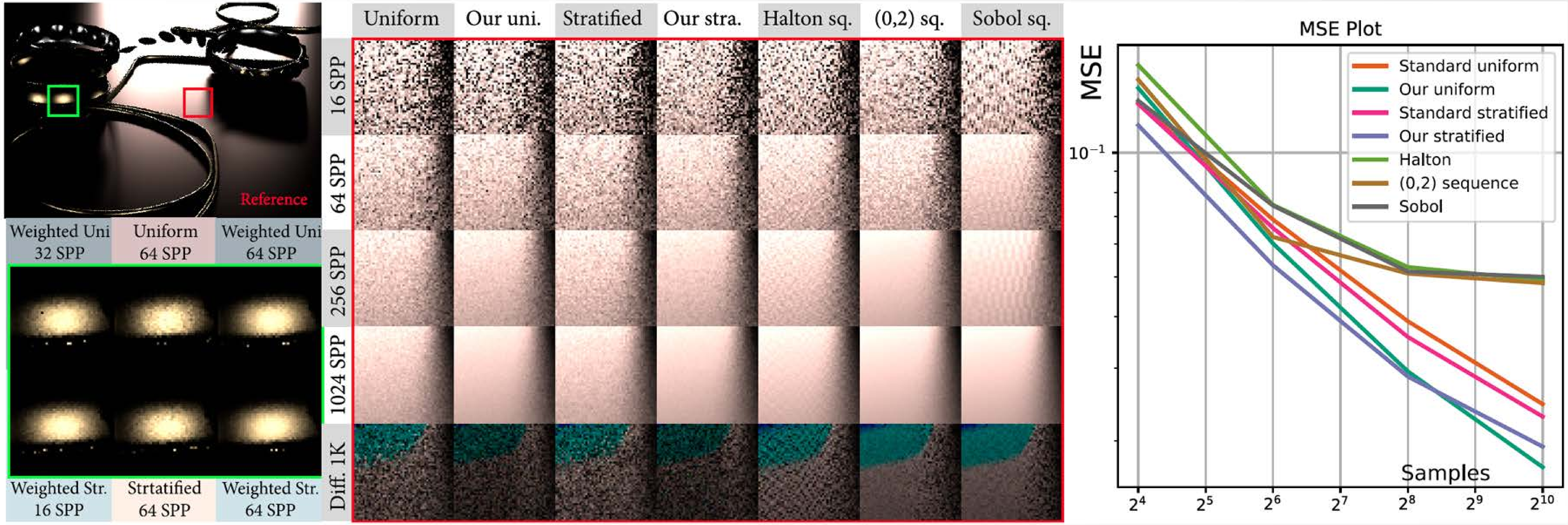
Results


MSE Plot
Standard uniform
Our uniform
Standard stratified
Our stratified
Halton
(0,2) sequence
Sobol
MSE
10^{-2}
Samples
2^4
2^5
2^6
2^7
2^8
2^9
2^{10}


Weighted Uni 128 SPP
Uniform 256 SPP
Weighted Uni 256 SPP
(0,2) seq. 64 SPP
Stratified 256 SPP
Weighted Str. 256 SPP


Weighted Uni 128 SPP
Uniform 256 SPP
Weighted Uni 256 SPP
Weighted Str. 128 SPP
Stratified 256 SPP
Sobol seq. 256 SPP

Results


Reference
Weighted Uni 32 SPP
Uniform 64 SPP
Weighted Uni 64 SPP
Weighted Str. 16 SPP
Strtatified 64 SPP
Weighted Str. 64 SPP
Uniform
Our uni.
Stratified
Our stra.
Halton sq.
(0,2) sq.
Sobol sq.
16 SPP
64 SPP
256 SPP
1024 SPP
Diff. 1K
MSE Plot
MSE
Samples
Standard uniform
Our uniform
Standard stratified
Our stratified
Halton
(0,2) sequence
Sobol
10^{-1}
2^4 2^5 2^6 2^7 2^8 2^9 2^{10}

Implementation Details

Precomputing samples and weights

- Small runtime overhead
- Reusing across render tiles

Limitations

Limited in dimensionality

- Voronoi tesselation expensive >3D

Only unbiased with 1D uniform samples

Future Work

Extend unbiased solution to higher dimension

Extend unbiased solution to non-uniform samples

Conclusion

Simple reweighting scheme for Monte Carlo integration

Unbiased 1D solution for uniform samples and piece-wise non-uniform

Effective and efficient

Robust comparing against Quasi Monte Carlo

Thank you!