Supplemental Material for Geometric Sample Reweighting for Monte Carlo Integration

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1. Introduction

In this document we provide additional experiments demonstrating the numerical performance and applicability of our estimator.

2. Special Functions

We apply our geometric sample reweighting to solving periodic functions, Dirac-delta functions, periodic step functions and non-periodic step functions. We choose functions that are typically common and difficult to integrate for MC.

The functions are given in the Tab. 1. The MSE plots are shown in Fig. 1. As can be seen from the plots, our estimator robustly handles different types of functions, despite the discontinuities and periodic properties. In all cases our estimator converges faster.

Type	Function
Periodic	$f(x) = \sin(2\pi x - 0.123456)$
Dirac-delta	$f(x) = \begin{cases} 50 & 0.79 < x < 0.81 \\ 0 & \text{otherwise} \end{cases}$
Periodic step	$f(x) = \begin{cases} 0 & 0.05 < x < 0.95 \\ 1 & \text{otherwise} \end{cases}$
Non-periodic step	$f(x) = \begin{cases} 0.1 & x \leq 0.1 \\ 0.2 & 0.1 < x \leq 0.2 \\ 0.3 & 0.2 < x \leq 0.3 \\ 0.4 & 0.3 < x \leq 0.4 \\ 0.5 & 0.4 < x \leq 0.5 \\ 0.6 & 0.5 < x \leq 0.6 \\ 0.7 & 0.6 < x \leq 0.7 \\ 0.8 & 0.7 < x \leq 0.8 \\ 0.9 & 0.8 < x \leq 0.9 \\ 1.0 & \text{otherwise} \end{cases}$

Table 1: *The functions we used for different types of characteristic.*

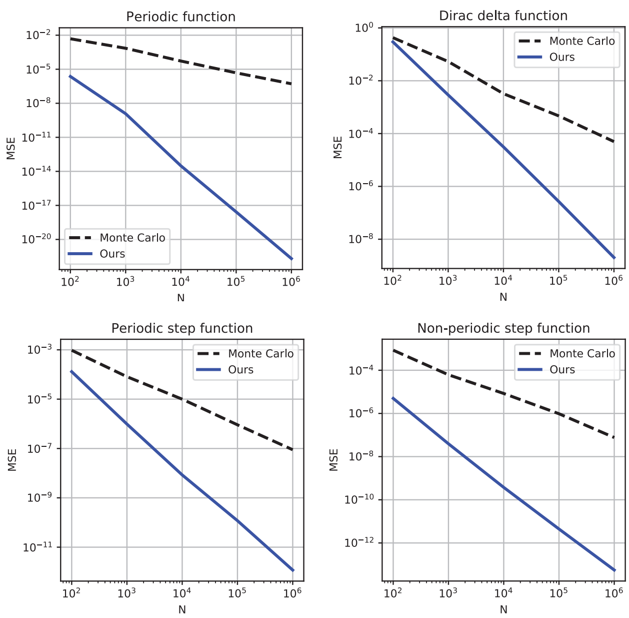

Figure 1: *MSE plot of applying our method to four kinds of functions.*

3. Polynomial Functions

To show the robustness of our estimator we tested a large collection of functions, here we, we give the numerical performance of our estimator solving 30 polynomial integrals with random coefficients. The function coefficients are given in Tab. 2. The MSE plots are given in Fig. 2.

4. Plot of Function g(x)

We plot for multiple N values the function g(x) in the paper in Fig. 3.

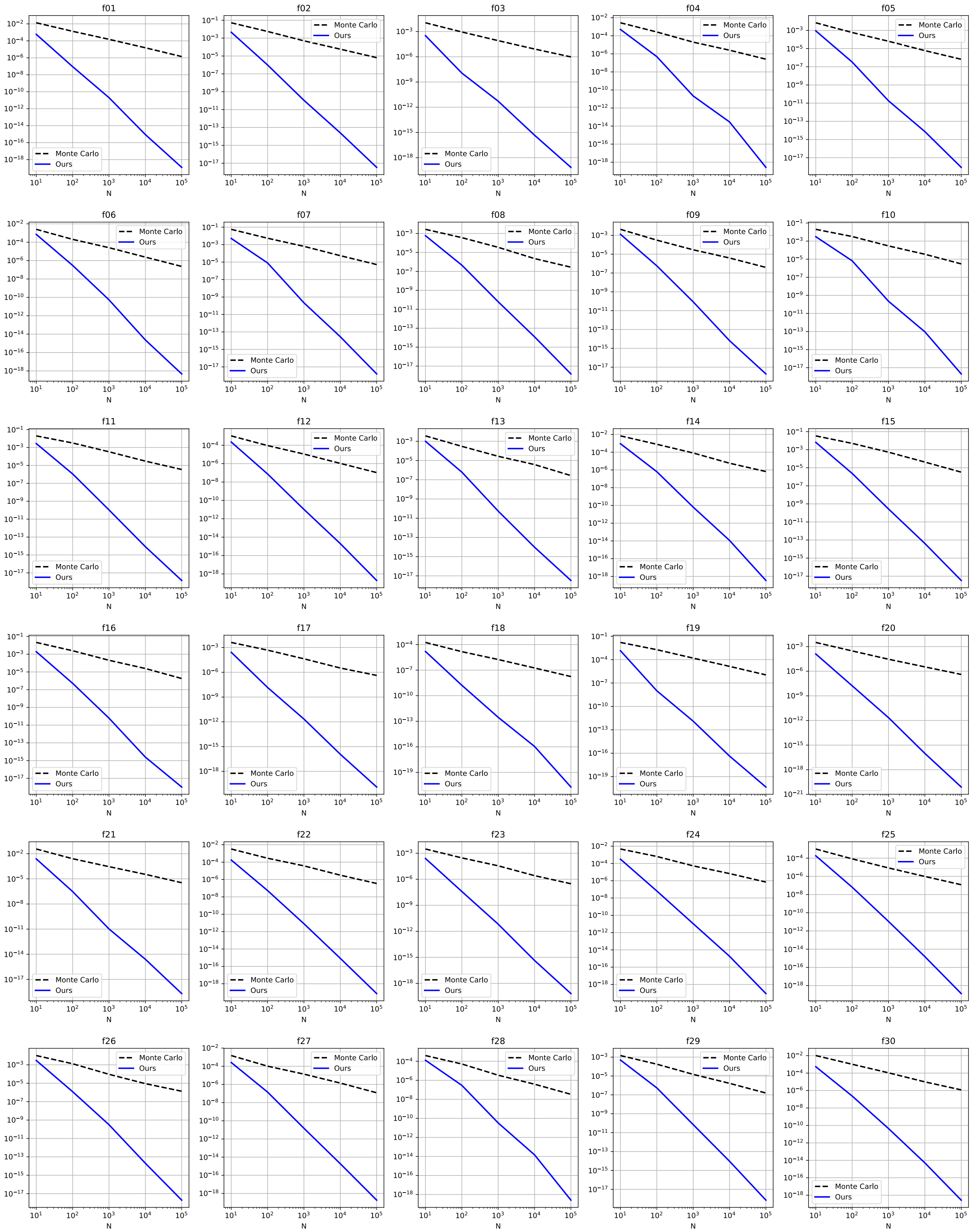

Figure 2: *MSE plot of applying our method on* 30 *random polynomial integrals.*

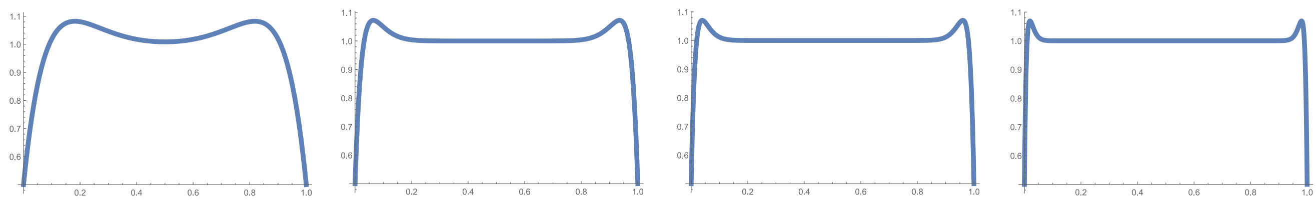

Figure 3: *Plot of function g(x) for $N = 10$, $N = 30$, $N = 50$ and $N = 100$. As can be seen from the plots, for most part of the domain, function g(x) converges to 1 as N increases.*

	$f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$					
	a	b	c	d	e	f
f01	0.1270	-0.0796	-0.0607	-0.2777	-0.9434	0.1794
f02	0.2558	0.9709	0.3217	0.4119	0.6448	-0.2426
f03	-0.7817	0.1987	0.7887	-0.1083	0.7784	-0.2332
f04	0.7744	-0.2005	0.5477	-0.8255	0.3722	0.8094
f05	-0.6722	-0.1236	-0.6632	0.7375	-0.2966	0.1440
f06	-0.5852	-0.0703	-0.4140	0.6962	-0.3085	0.8526
f07	0.7614	0.8993	0.2737	0.1318	0.6948	0.7587
f08	-0.7105	-0.2057	-0.6851	0.8309	0.7212	-0.2506
f09	-0.5636	-0.6467	-0.7671	0.9434	0.8707	-0.3383
f10	-0.8551	-0.7562	-0.6061	0.9055	-0.8138	0.8493
f11	-0.7370	0.7543	-0.8533	-0.5964	-0.5153	-0.2185
f12	0.5759	-0.8744	-0.1685	-0.5950	0.9866	0.0288
f13	0.3687	0.4285	0.6883	-0.4559	-0.4845	-0.2279
f14	0.8275	-0.2912	0.0001	-0.0850	0.6052	-0.4128
f15	0.4258	0.6358	0.5405	0.2875	0.6074	0.8527
f16	-0.4618	-0.6037	0.0678	-0.5086	-0.2342	-0.0464
f17	0.0021	0.6122	-0.6367	-0.2097	-0.4176	-0.2965
f18	0.4714	-0.5152	-0.3126	0.1877	0.2786	0.6883
f19	0.9205	-0.1603	-0.6306	-0.9718	-0.3250	0.2473
f20	-0.5333	0.7581	-0.5389	0.3670	0.4817	0.1416
f21	0.6755	-0.9932	0.2748	-0.9123	-0.9967	0.7738
f22	0.9415	-0.9712	-0.0860	0.1380	0.6473	-0.3570
f23	-0.4338	0.8926	-0.0842	-0.6906	0.9707	0.0057
f24	-0.0539	0.0423	-0.2354	-0.0008	-0.6465	-0.0221
f25	-0.2629	0.9509	-0.2101	0.3154	-0.5717	-0.3693
f26	0.7111	0.8524	-0.1501	0.5249	-0.7708	-0.5785
f27	0.6856	-0.8118	0.6545	0.3048	-0.9819	0.4253
f28	-0.6248	-0.1219	0.1694	0.4370	-0.0347	-0.0560
f29	0.9477	0.5040	-0.6288	-0.2215	-0.3869	-0.6021
f30	-0.7930	0.6597	-0.2526	-0.0821	-0.8163	0.0488

Table 2: *The random coefficients used for the polynomial functions.*

5. Periodically Augmented Sample Set

The derivation of the expectation of periodically augmented sample set is given at the end of the document.

6. Conclusion

As can be seen from the plots above, our estimator robustly handles different kind of functions while consistently delivering a smuch faster convergence rate. This underlines that our estimator can be applied to many types of Monte Carlo integration problems.

$$
\begin{aligned}
\mathbb{E}\left[\hat{I}_P\right] &= N! \int_0^1 \int_{x_1}^1 \cdots \int_{x_{N-1}}^1 \left[\frac{x_2+1-x_N}{2} f(x_1) + \sum_{i=2}^{N-1} \frac{x_{i+1}-x_{i-1}}{2} f(x_i) + \left(\frac{1+x_1-x_{N-1}}{2}\right) f(x_N)\right] \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_N \\
&= \frac{N!}{2} \int_0^1 \int_{x_1}^1 \cdots \int_{x_{N-1}}^1 f(x_1) \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_N + \frac{N!}{2} \int_0^1 \int_{x_1}^1 \cdots \int_{x_{N-1}}^1 x_2 f(x_1) \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_N - \\
&\quad \frac{N!}{2} \int_0^1 \int_{x_1}^1 \cdots \int_{x_{N-1}}^1 x_N f(x_1) \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_N + \frac{N!}{2} \sum_{i=2}^{N} \left\{ \left[x_{i+1} f(x_i) - x_{i-1} f(x_i)\right] \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_N \right\} + \\
&\quad \frac{N!}{2} \int_0^1 \int_{x_1}^1 \cdots \int_{x_{N-1}}^1 f(x_N) \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_N + \frac{N!}{2} \int_0^1 \int_{x_1}^1 \cdots \int_{x_{N-1}}^1 x_1 f(x_N) \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_N - \\
&\quad \frac{N!}{2} \int_0^1 \int_{x_1}^1 \cdots \int_{x_{N-1}}^1 x_{N-1} f(x_N) \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_N \\
&= \frac{N!}{2} \int_0^1 \frac{(1-x)^{N-1}}{(N-1)!} f(x) \mathrm{d}x + \frac{N!}{2} \int_0^1 \frac{(1-x)^{N-1}[(N-1)x+1]}{N!} f(x) \mathrm{d}x - \frac{N!}{2} \int_0^1 \frac{(-1)^{N-2}(x-1)^{N-2}(x+N-2)}{(N-1)!} f(x) \mathrm{d}x + \\
&\quad \frac{N!}{2} \int_0^1 \frac{[N(x-1)-2x](x-1)x^N + x\left\{Nx+2x-1+(x-1)^N\left[Nx-2x+2-(1-x)^{-N}(Nx+1)\right]\right\}}{(x-1)N!} f(x) \mathrm{d}x + \\
&\quad \frac{N!}{2} \int_0^1 \frac{x^{N-1}}{(N-1)!} f(x) \mathrm{d}x + \frac{N!}{2} \int_0^1 \frac{x^N}{N!} f(x) \mathrm{d}x - \frac{N!}{2} \int_0^1 \frac{x^N}{N(N-2)!} f(x) \mathrm{d}x \\
&= \int_0^1 \frac{(-1)^{1+n} n(x-1)^n(n+x-2) + (x-1)\left\{nx+2x-1-(1-x)^n\left[n+x-1+(1/(1-x))^n(nx+1)\right]\right\}}{2(x-1)^2} f(x) \mathrm{d}x
\end{aligned}
$$